ALGEBRA THROUGH THE LENS OF FUNCTIONS

JUNIOR CERTIFICATE HIGHER LEVEL

PART 1 OF 2

Introduction

To Teachers

Algebra Through the Lens of Functions Part 1 and 2 have been designed by the Maths Development Team for use by teachers of mathematics. *Algebra Through the Lens of Functions* Part 1 of 2 should be used prior to engaging with Part 2. Both part 1 and part 2 treat the art of teaching algebra through the lens of functions through a series of units. The material contained in the document is suitable for all levels and abilities but is particularly suited to Junior Certificate Higher Level. It includes a discussion of activities, tasks and the formation of connections suitable for classroom use. When necessary, required subject matter content is covered as well. Both Part 1 and 2 were written in response to the many teachers who attended continuing professional development courses given by the authors and were unable to find material in a single convenient source. The authors collaborated with the Maths Inspectorate of the Department of Education & Skills to provide a collection of activities and strategies for Junior Certificate mathematics classes.

Activity Book

Visualising Patterns for Linear Relationships and *Visualising Patterns for Quadratic Relationships* are activity workbooks which can be used to supplement the text in Part 1 and Part 2 respectively. *Visualising Patterns for Linear Relationships* is an activity workbook which can be used to supplement this text. It contains ten patterns that may be used for each unit of the text. *Visualising Patterns for Linear Relationships* can be downloaded from here. Larger versions of the images are available in PDF form here and as an interactive PowerPoint here.

Organisation, Format & Special Features of the Units

Units 1 to 10 deal with linear relationships. Unit 1 introduces linear relationships through the visual stimulus of the "Dots Activity". The need for the skills of representing variables using letters and substitution into linear expressions should become evident to students during this Unit. At this stage, students will recognise that substitution into an expression is a more efficient strategy than continuing a table when finding outputs for a given input. The use of visual patterns as a stimulus is continued in Unit 2 when solving linear equations is introduced. During Unit 2 students will recognise that solving an equation is a more efficient strategy than continuing a table when the output is known and the corresponding input is required. Different students may see the same visual pattern slightly differently and the skills of visualising, simplifying and factorising expressions are thus required. These are explored in Units 3 and 4 and it will become evident that as visual patterns become more complex, the range and variety of expressions that can be used to represent them increases. Unit 5 focuses on the Money Box Activity and examines the equation $y = mx + c$ in a variety of ways. This is an extremely important Unit as all other functions can subsequently be compared to linear functions. Unit 6 begins the process of rearranging formulae. Again, students should understand the benefits of the skill before learning it. For example, it may be more efficient to rearrange an equation initially and then use substitution into the rearranged equation rather than solving the equation repeatedly to find outputs for given inputs. Rearranging formulae for non-linear relationships will be addressed later in the document. The efficiency of adding functions to solve problems is looked at in Unit 7, which also motivates the skill of simplifying the sum of like terms. Comparing linear functions and solving simultaneous equations and inequalities is introduced using the growing patterns of four sunflowers in Unit 8. The students will solve their first simultaneous equation problem using a table and graph and then engage with the problem using algebra. The need for the algebraic method is that the use of tables and graphs becomes inefficient and inaccurate if the solution has large or non-integer values. Unit 9 introduces the elimination method for solving simultaneous equations through a problem that can also be solved using trial and improvement. A similar problem with much larger numbers could be shown to students to illustrate the need for the greater efficiency of the elimination method. Unit 10 looks at algebraic expressions and algebraic equations that contain fractions.

Contents

General Overview

The central idea in this document is that it might be better for students to understand why a skill is required before learning it. In addition to endeavouring to make functions the focus of the document, multiple representations are used throughout and some links back to what can be done when students are studying Number to help students understand algebraic concepts are made. Therefore, the learning outcomes of both Strands 4 and 5 are included in this document. The intention is that by the time students are "finished" their work on Algebra that they are also "finished" their work on Functions. The remainder of this overview looks at (a) the flow of some Units in this document, (b) the rationale for the positioning of some algebraic skills and (c) key features of functions.

(a) The Flow of Some Units in this Document

Broadly speaking the flow below is used in many Units in this document to show students the need for the algebraic skills they are about to learn prior to learning the skill. Throughout the document multiple solution strategies are used. In many cases students can find the answer to various questions by analysing a table or by interpreting a graph before engaging with the problem using an algebraic method. This means that the algebraic method should make sense to the students as they already know what the answer should be. In many cases too the need for an algebraic method is obvious as other methods become be too tedious when the numbers are large or prove inaccurate when the answer is not a whole number. Students can bring their own thinking to many of the problems before they are introduced to the formal algebraic approach. By doing so, the students get a greater sense of what they are doing and why they are doing it, can recognise the value of thoughtful engagement with problem solving and appreciate that algebraic techniques offer them incredibly powerful ways of tackling problems. Students should then have a greater appreciation of where algebra fits into mathematics as a whole and how understanding algebraic relationships and techniques is worthwhile.

1. Students engage with a problem

Students engage with a problem through (i) whole class discussion led by teacher questioning, (ii) working in groups, (iii) individual work or (iv) a combination of some or all of the above. Following this, the students implement strategies such as drawing diagrams, analysing a table, trial and improvement or interpreting a graph. The effectiveness of these strategies are then compared by the students through discussion.

2. Students see the need for a new strategy

The students engage with a follow-on problem where the limitations of earlier strategies become apparent and the need for a new strategy is obvious.

3. Students are guided by the teacher to learn the new strategy

Students are guided by the teacher to learn the new strategy. In this document the new strategy will always incorporate algebraic solutions.

4. Students compare the new strategy with the previous strategies

Students compare the new strategy with the previous ones to see the advantages of the algebraic approach and to recognise that algebraic solutions can be checked by using other methods.

(b) Rationale for the position of some algebraic skills

It is acknowledged that students may have prior knowledge (of using the laws of indices when multiplying numbers with a common base or using the distributive law when working in number, for example), but it is envisaged that moving from expanding expressions of the type $3(x+2)$ to more complex examples like $x(x+2)$, $(x+2)(x+3)$ or $(x+2)(x^2+3x+4)$ is delayed until the need for such expansions is obvious. For this reason, work on linear relationships is treated separately from that on quadratics and cubics. This also brings the additional benefit that when a student looks at $(x+2)(x+3)$ for the first time they will see it from a number of perspectives and not just as an algebraic skill.

Once a skill is learned in one Unit it can then be used in all subsequent Units. For example, substitution is learned in the "Dots Activity" in Unit 1 and while it might not be explicit in the Units that follow it should be seen as a key skill in developing understanding in those Units too.

(c) Key Features of Functions

Throughout the document the key features of functions are referred to. These features can be used when analysing functions so students can use the same criteria for analysing the various functions they encounter as they progress from first year through to sixth year.
The key features of functions are:
1. The domain and range
2. Where the graph of the function meets the axes.
3. What is constant and what varies in the function?
4. The behaviour of the graph of the function
5. The rate of change of the function

Note: Average rates of change can be used to begin the discussion about how rates of change can be positive, negative or zero and how this can be used to decide if the function is increasing, decreasing or neither. This work can be continued at senior cycle when the slope of the tangent to the function can be dealt with.

How to use this document

Throughout the document there are hyperlinks to useful resources, for example, booklets of visual patterns, matching activities, Teacher Resource Booklets from the various workshops and Teaching and Learning Plans. Clicking on a hyperlink will bring you to the resource.
The document also contains boxes entitled "Number Work" which provide ideas that should be used when students are studying number, per se, but that can also help students see properties of number that are important for algebra. They are included in this document so that they can be revisited when students encounter the related concept in algebra.
Each Unit contains a number of Sample Problems that can be used with students.

Unit 1: A Linear Relationship – "Dots Activity"

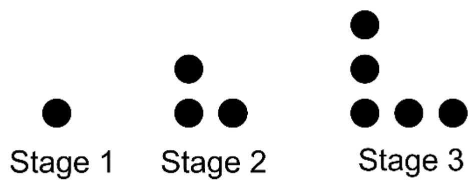
In this Unit students will:

- express a relationship in words
- develop and use their own generalising strategies and ideas and consider those of others
- use letters to represent variables

This Unit uses one question that has a visual pattern as its stimulus. It will be referred to in this document as the "Dots Activity". It is not envisaged that everything in this Unit will be done by exploring just one question. Indeed a number of questions could be used to introduce and explore the concepts that are outlined in this Unit. To facilitate the integration of this approach in lessons for this Unit and other Units, a booklet of linear patterns is available here.

Sample Problem

Find a relationship between the stage number and the number of dots.



It might be best to introduce this activity to first-year students as a "patterns puzzle" rather than saying "linear relationships". The phrase "linear relationships" could be used by the end of this task or other similar tasks if the graph representing the relationship and utilising a dotted line to illustrate the linear pattern is drawn.

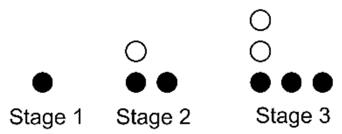
Students should be encouraged to express the relationship in words.

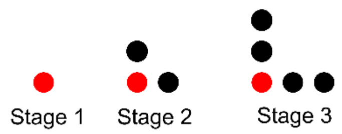
Prompting students to draw the first four stages (with the option of using more than one colour) themselves and to draw the next stage in the pattern might help students understand the pattern. Questions like "Where can you see four dots in stage four" can also be helpful.

Enabling students to use a range of expressions to explain their reasoning is important. The expressions they may use could include: double, twice, two times, less, minus, subtract, reduce, up, across, vertical, horizontal, bottom, side, shared, double-counted, etc.

Students might say that say "there are an odd number of dots in each stage". This will require further exploration as, while the outputs are certainly odd numbers, the pattern is a particular sequence of odd numbers.

Students might say "you add two dots each time" or "start at one and add two dots each time" or have spotted a pattern i.e. they have generalised. These examples focus on the number of dots but they have not related the number of dots to the stage number. These expressions are not the most useful for working out (i) how many dots are in the 100^{th} stage or (ii) which stage has 83 dots. "Twice the stage number less one" and "the stage number + the stage number less one" are more useful for this purpose.

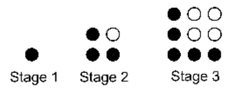




Once students have expressed the relationships in words they can be encouraged to use letters as an efficient way of communicating the relationship. The skill of being able to express a relationship in symbols should not replace verbalising the relationship as both are useful.

Expressions like "twice s–1", "s+s–1" and "1+s–1+s–1" are more useful than "start at one and add two dots each time" for working out how many dots are in the 100^{th} stage. Students can benefit from explaining their reasoning and understanding other students' different ways of expressing the relationship. It is also worth pointing out that the expressions "twice s–1", "s+s–1" and "1+s–1+s–1" are all equivalent.

$s^2-(s-1)^2$ is possible but might not be the most important thing at this stage with a first-year class.



Next, Near, Far, Any

The phrase “next, near, far, any” is a useful one to keep in mind when working with visual patterns. Progressing through “next”, “near”, “far”, “any” should enable students to see the advantages of generalising.

In this context “next” is the next (or 5th) stage. There are 9 dots in the 5th stage. This answer can be found using many strategies, for example, drawing the stage or continuing a table.

In this context “near” is the something like the 10th stage. There are 19 dots in the 10th stage. This answer can be found using many strategies, for example, drawing it or continuing a table. However, drawing all the intervening stages is time consuming.

In this context “far” is something like the 100th stage. There are 199 dots in the 100th stage. Drawing the intervening stages or continuing the table to the 100th stage will be too time consuming. Asking how many dots there are in a far stage encourages students to choose to generalise themselves.

Asking students how many dots there are in “any” stage is explicitly asking them to generalise. It is hoped that they would have chosen to generalise when asked how many dots are there in a “far” stage.

Notes:

1. In addressing questions of the type “Find a relationship between the stage number and the number of dots”. It is important to emphasise that a relationship should contain; =, <, >, ≤, or ≥ (or a verbalised version of these). For example, students might say the relationship is $s+s-1$, where s is the stage number. Strictly speaking, the relationship in the “Dots Activity” should resemble $t=s+s-1$, where s is the stage number and t is the total number of dots in each stage.
2. While the students will come up with an **expression** it should take the form of an **equation**, for example, Number of Dots in each stage = Twice(s)–1
3. If the students do not organise their thoughts with a table then they should be shown a table of inputs and outputs. Plotting points is part of the Common Introductory Course (CIC) and should be addressed before introducing the students to algebra. This will enable students to plot each input and corresponding output as a set of points and to see the relationship in another representation. Information and communication technology (ICT) can be used to create many sets of inputs and outputs and plot the sets of points.
4. **Variables:** s, which represents the stage number in the relationship is a variable, as the stage number varies. t, which represents the total number of dots in each stage is also a variable. This topic opens up the possibility of in-depth collaboration with the science department at some stage as the concepts of independent and dependent variables will be met by those students studying junior certificate science.
5. **Constants:** The “–1” in the expressions or relationships is constant.

A Challenge for Homework

The following problem could be posed to students: "Which stage contains 47 dots?"

Formally solving problems like this will be the focus of the next Unit but it is useful for students to engage with this type of problem in an informal way before engaging with it more formally.

Trial and Improvement: Some students in the class should be able solve the problem by using trial and improvement. For example, stage 20 has 20+20–1 dots which is too few, stage 30 has way too many dots, stage 25 is getting very close and stage 24 has 24+24–1 dots which is just right.

Using a Table: Some students could make out a table and continue the table to stage 24.

Interpreting a Graph: Some students could make out a graph of points and see that the output is 47 when the input is 24.

"Undoing the Relationship": Some students might try to "undo the relationship". They might do this incorrectly by reducing the 47 by one and dividing by two to get stage 23 or they might do it correctly by increasing the 47 by one and dividing by two to get stage 24. Both approaches should form the focus of subsequent whole-class discussion.

Note: The description above focuses on one part of the relationship, the 47. It would be beneficial to share with students a solution that encapsulates the whole relationship.

Twice the unknown stage number minus one is 47.

This means twice the unknown stage number is 48.

Therefore the unknown stage number is 24.

Unit 2: Solving Linear Equations

In this Unit students will:

- use the terminology *input* and *output*
- solve a linear equation of the form $ax + b = c$ algebraically
- understand the difference between a variable and an unknown
- develop and use their own generalising strategies and ideas and consider those of others
- use letters to represent numbers

Sample Problem

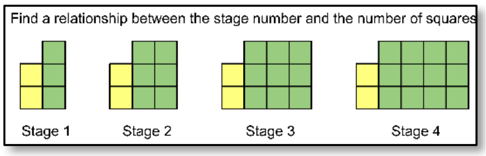


Figure 1

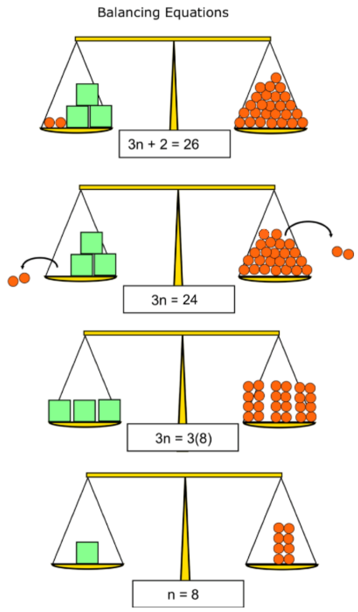
(i) Find a relationship between the stage number and the number of squares.
(ii) Which stage has 26 squares?

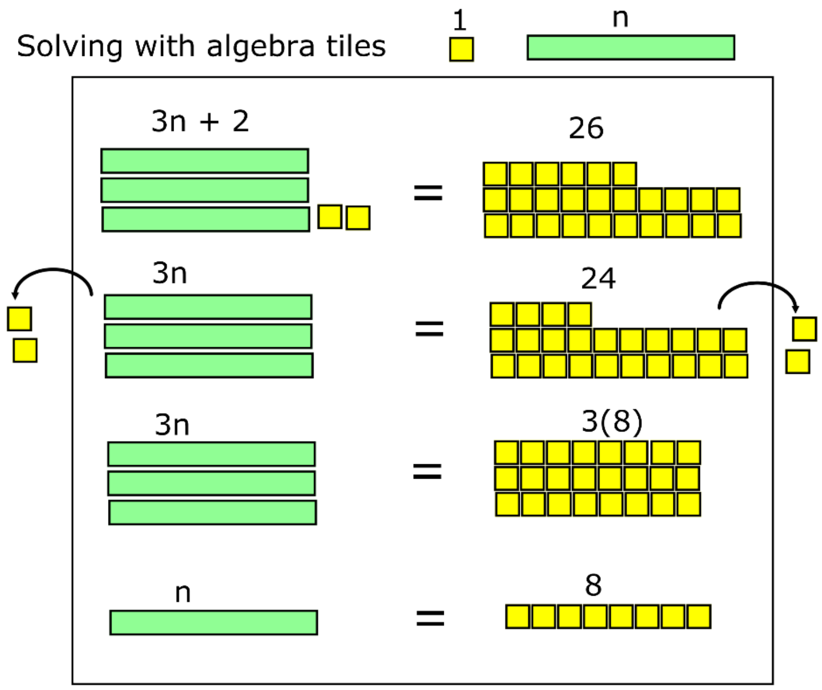
When students are ready to solve linear equations, a question similar to the "Challenge for Homework" from the previous Unit should be presented to them. The visual pattern could be one where the expression representing the pattern is $3n + 2$ (see Figure 1 above) and the stage containing 26 squares is required? Students should be asked to explain their solution(s). If a student says "take away 2 and divide by 3" write "take away 2 and divide by 3" and "$3n + 2 = 26$" on the board and implement the student's strategy using the concept of inverse operations and doing the same thing to both sides. Stabilisers should now be introduced to formalise the process.
Three possible teaching approaches are illustrated below. The first approach is based on the stabilisers' method from the Teaching and Learning Plan that can be found here. The second and third approaches visualise the process through preserving balance.

	$3n + 2 = 26$	
-2	$3n + 2 - 2 = 26 - 2$	-2
	$3n = 24$	
$\div 3$	$\frac{3n}{3} = \frac{24}{3}$	$\div 3$
	$n = 8$	

	$3n + 2 = 26$	
-2		-2
	$3n = 24$	
$\div 3$		$\div 3$
	$n = 8$	

The process above could be described using the following three sentences:
Three times the unknown stage number plus 2 is 26.
This means three times the unknown stage number is 24.
Therefore the unknown stage number is 8.
This logical progression may also be thought of in terms of isolating the unknown.





Notes

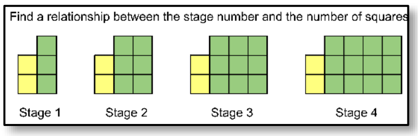
1. A more difficult equation might also be introduced to justify the need for the procedure.
2. **Unknown:** There is an opportunity here to talk about the difference between n as a variable in the expression $3n + 2$ compared to the relationship $t = 3n + 2$ where n is an unknown and t is a variable. A table or a graph can also be used to show that n can take on many values but when we are solving we are only interested in the value(s) that satisfy the equation in question.
3. It is a good idea to use a table and/or a graph the first time students formally solve an equation so that they can see the solution in these representations and can understand what they have just done.

The "Challenge for Homework" from the previous Unit could be returned to now with the new knowledge and thus solving the equation $2n - 1 = 47$ formally.

Inputs and Outputs and Informally Looking at Some Concepts of Functions

The phrases *input* and *output* can be used by both the teacher and students from quite an early stage for activities similar to those in Units 1 and 2.
Substitution is used to find the output for any given input.
Solving equations provides an input from a given output.
Independent and dependent variables can also be discussed. It is worth noting that the concepts of independent and dependent variables will be met by those students studying junior certificate science.

Questioning can be used in laying the foundation for the concepts of *domain*, *co-domain* and *range*. Three examples of the type of questioning that could be considered are given below. This type of questioning could also be used to revise the different number systems and make a link to discrete and continuous data. The questions are designed to lay the foundation for the concepts of *domain*, *co-domain* and *range*. Once the concept is understood the terminology can be introduced at the appropriate time. This could be at any stage from first to third year.



<u>Question 1. How could the inputs be described?</u>
Possible student answers might include:
1. The inputs are positive whole numbers.
2. The natural numbers
3. {1, 2, 3, 4 …}

The concept of the *domain* is introduced here in a meaningful way as there is a context to which to the concept can be linked. Students should be able to see that neither stage –32 nor stage 2¾ make any sense.
Over time it might seem unwieldy to keep saying the set of inputs so there becomes a need for the term *domain*. When students are ready, they can be told that the set of inputs in a relationship is called the *domain* and this should be then used interchangeably with the phrase "set of inputs".

<u>Question 2. How might the outputs be described?</u>
The students might respond that the outputs are all positive whole numbers.
While this answer is broadly correct it lacks rigour and opens the possibility of discussing the idea of the *co-domain*.
The outputs are indeed all positive numbers, but all the positive whole numbers are not included in the set.

<u>Question 3. Can you define the outputs more specifically?</u>
The students may describe the output as the set {5, 8, 11…} or the whole numbers starting with five and increasing constantly by three thereafter, etc.
The discussion that ensues allows for the definition of the *range*, as the set of outputs, to be introduced and explored

Thinking ahead
Examples 1 and 2, below, indicate that this type of approach can be revisited for continuous linear and quadratic relationships:
Example 1: A sunflower with a starting height of 2 cm and a growth rate of 3 cm per day could be modelled by a function that has a domain of $x \in \mathbb{R}$, $x \geq 0$, a co-domain of $y \in \mathbb{R}$, $y \geq 0$ and a range of $y \in \mathbb{R}$, $y \geq 2$.
Example 2: A quadratic like $x^2 - 9$ could have a domain of $\mathbb{R}$, a co-domain of $\mathbb{R}$ and a range of $y \in \mathbb{R}$, $y \geq -9$

Unit 3: Linear Expressions in Multiple Representations

In this Unit students will:

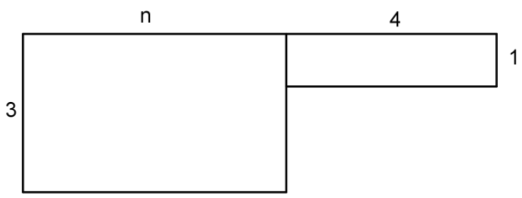
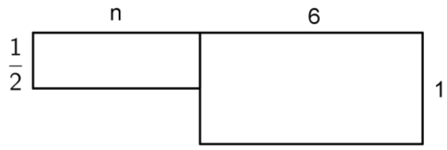
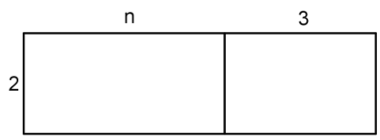
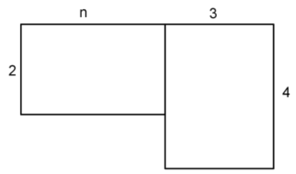
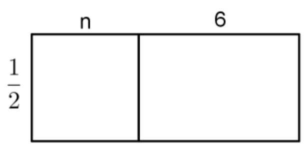
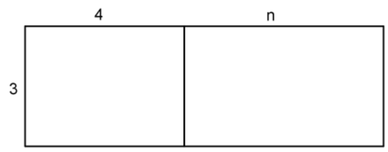
- match representations of linear algebraic expressions
- draw an area representation when given a linear expression expressed in words or symbols
- visualise like and unlike terms
- visualise the distributive law
- use letters to represent variables
- visualise some transformational activities, for example collecting like terms and expanding
- recognise when two algebraic expressions are equivalent

There are two central activities to this Unit: **A.** Matching and **B.** Drawing. The activities aim to help students visualise the ***distributive law for algebraic expressions***, like and unlike terms, and equivalent expressions.

A. Matching Activity

The aim of this activity is to provide students with the opportunity to visualise algebraic expressions in multiple representations and visualise the ***distributive law for algebraic expressions***. Some students make errors when expanding $2(n + 3)$ and think it is equivalent to $2n + 3$ or $2n + 5$. The activity below helps students see the equivalence of $2(n + 3)$ and $2n + 6$. Following this activity (and the drawing one immediately following it) students should practice expanding expressions of the form $a(x + b)$.

This activity uses three representations (Words, Symbols and Area). It is possible to include further representations, for example, a table or a graph. In order to make this activity challenging for more able students, solutions that include fractions are also included. The activity focuses on linear expressions for now. A similar activity focusing on quadratic expressions will be used in a later Unit. The matching activity for linear expressions can be found here.

A1 	**A4** 
A2 	**A5** 
A3 	**A6** 

E7 $3n + 4$	**E4** $\frac{n}{2} + 6$
E8 $2n + 6$	**E5** $2n + 12$

E1 $2(n + 3)$	**E3** $3n + 12$
E2 $\frac{n + 6}{2}$	**E6** $3(n + 4)$

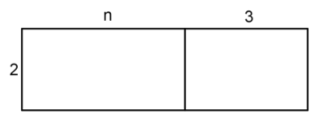
W3 Multiply n by three, then add 4.	**W8** Divide n by two, then add 6.
W4 Multiply n by two, then add 6.	**W2** Multiply n by two, then add 12.
W1 Add three to n, then multiply by two.	**W6** Multiply n by three, then add 12.
W7 Add six to n, then divide by two.	**W5** Add four to n, then multiply by three.

The solutions to the above activity are:

A1	E7	W3
A2	E1, E8	W4, W1
A3	E2	W7
A4	E4	W8
A5	E5	W2
A6	E3, E6	W6, W5

An extension to this activity would be to ask students to create their own expressions and represent them using words, area etc.

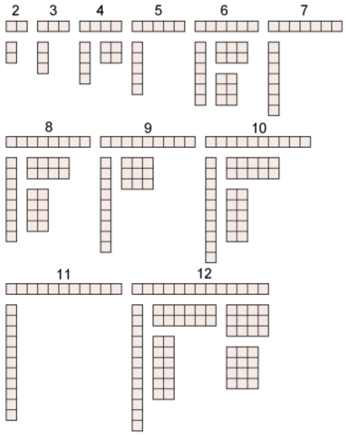
Note: Matching $2(n + 3)$ and $2n + 6$ with the diagram below could be the first time students will have seen the distributive law containing variables.



Number Work

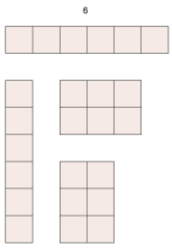
Understanding factors incorporating prime, composite and square numbers would all help with the previous and later activities, including, for example, simple factorising, factorising by grouping and factorising quadratics.

By building all the possible rectangles from whole numbers of unit squares (for example 2, 3, 4, 5, 6, 7, etc.) students will see that some numbers only have a very limited choice in how a rectangle can be built, for example, 2, 3, 5, 7 etc. (*prime numbers*).



Some numbers have more choice in how a rectangle can be built, for example, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20 etc. (*composite numbers*).

Some of this second set of numbers can be arranged into a square, for example, 4, 9, 16, 25 etc. (*square numbers*).

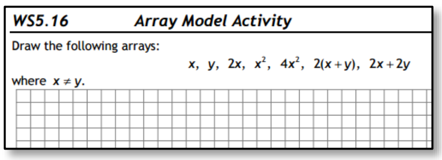


A discussion can be had about how the number 6 can be represented as 3 groups of 2 and also as 2 groups of 3 to illustrate the commutative law.

B. Drawing

The aim of this activity is to visualise like and unlike terms, the distributive law and equivalent expressions.

A similar activity to this focussing on the quadratic, was used in the Workshop 5 Teacher Resource Booklet. This document, however, focuses on linear expressions for now.



WS5.16 ***Array Model Activity***

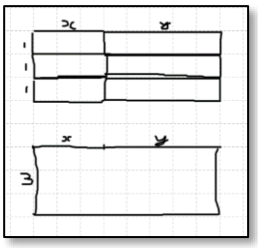
Draw the following arrays: x, y, $2x$, x^2, $4x^2$, $2(x+y)$, $2x+2y$

where $x \neq y$.

Students are asked to draw the arrays to represent: $x, y, 2x, 2y, 2x+2y, 2(x+y), 3x+3y, 3(x+y)$ where $x \neq y$.

Students should be encouraged to use squared paper for this activity.

Prompting students to see x as $1x$ or $+1x$ and y as $1y$ or $+1y$ may also prove useful.

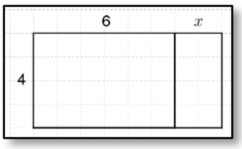


Number Work

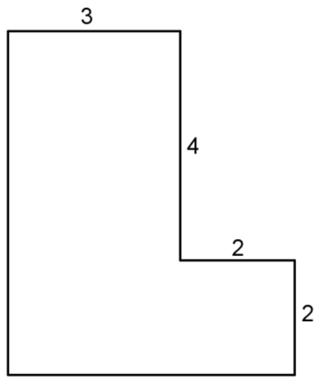
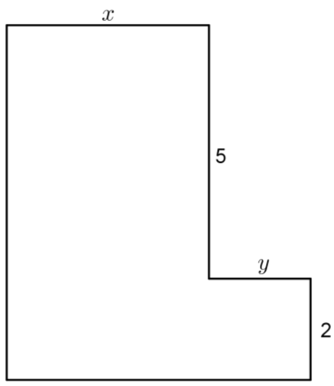
Using simple factoring and the distributive law when writing numerical expressions in many equivalent forms will help students with their understanding of writing algebraic expressions in equivalent forms.
8+8+8+8 can be written as 4(8), 32, 4(3+5), 4(3)+4(5) and as 12+20 and ideally students should be able to move freely between many of these representations. Area representations of all for these can also be drawn.
Seeing that 3(35)=3(30+5)=3(30)+3(5) will make it easier to understand that $3(x+y)=3x+3y$
Seeing like numbers being grouped together using factorisation and drawn as rectangles could also help students understand some of the above, for example, 2+2+2+2=4(2) which could help students see that $3x$ can be drawn as a rectangle with dimensions 3 and x, but it can also be drawn as 1 by $3x$ or as 1 by $(x+x+x)$.

<u>Additional Idea 1</u>
Students might be asked to multirepresent the following situation:
"A room has dimensions 6 m by 4m. If the longer side is extended by x metres what will the area of the room now be?"
Expression: $4(x+6)$ m^2
Words: 24 m^2 plus four times x m^2

Diagram:



<u>Additional Idea 2</u>
In order to improve their understanding of equivalent expressions, the students might be asked to express the areas of L-shapes that have numeric/variable sides in three different ways.

	**Top+Bottom** 3(4)+2(5) **Left+Right** 3(6)+2(2) **Total–Corner** 5(6)–4(2) **Note:** units squared should be used throughout.		**Top+Bottom** $5x+2(x+y)$ **Left+Right** $7x+2y$ **Total–Corner** $7(x+y)-5y$ **Note:** units squared should be used throughout.

Note: The special case of an L-shape which represents the difference of two squares will be examined later.

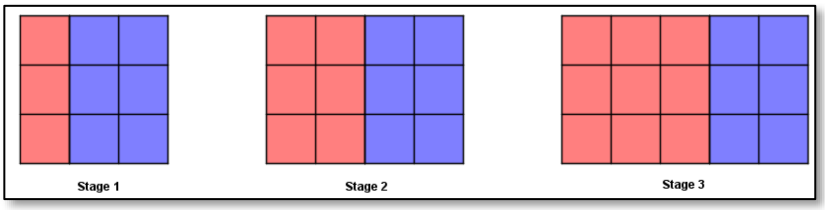
Unit 4: Algebraic Expression Skills

In this Unit students will:

- factorise
- rearrange areas into rectangular shapes so the area can be expressed as a product of factors
- draw an area representation when given a linear expression expressed in words or symbols
- visualise some transformational activities e.g. collecting like terms, simplifying expression, expanding and factoring.
- add and subtract simple algebraic expressions of the form $(ax + by + c) \pm (dx + ey + f)$

Some informal work on simplifying the sum and difference of like terms took place in previous units, for example, in the "Dots Activity" students may have come up with expressions such as "twice s–1", "s+s–1" and "1+s–1+s–1" are all equivalent. This Unit focusses on formally simplifying the sum and difference of like terms and simple factorising using areas of rectangular shapes. Students will sketch and understand expressions of the form $a(bx + c)$.

Sample Problem 1



(i) Describe the relationship between the stage number and the total number of tiles.
(ii) Describe the relationship between the stage number and the number of red tiles.

Prompting students to draw the first three stages themselves and to draw the next stage in the pattern might help students understand the pattern.
Questions like "Where can you see three tiles in stage three" can also help students.

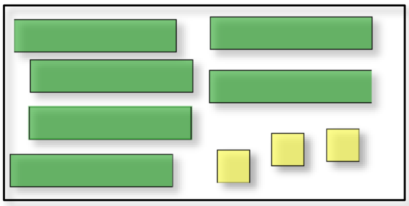
Colour is used so students can see the relationships in a variety of ways:
Blue
Some students will see two blue tiles in each row and add up three twos i.e. 2+2+2 = 6.
Other students will see two blue tiles in each row and see three rows and multiply three by two i.e. 3(2) = 6.
Other students will see a total of six blue tiles in each stage.
Red
Some students will see three red tiles in the first stage, six in the second stage and nine in the thirds stage meaning the total number of reds in each stage is three times the stage number i.e. $3x$.
Some students will see a stage number of red tiles in each of the three rows of the pattern i.e. $x + x + x$.
Total
The relationship between the total number of tiles, t and the stage number, x, can be written as $t = x + 2 + x + 2 + x + 2$ or as $t = 3(x + 2)$ or as $t = 3x + 6$.

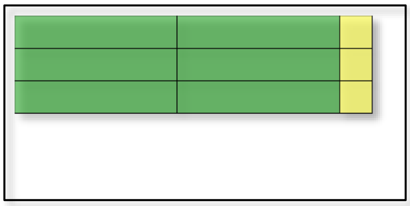
Notes:

1. A table and a graph could be used to represent the inputs and outputs at this stage. Alternatively, this could be deferred until the next Unit where these representations will be looked at in detail.
2. Many students, from their work with the order of operations in number, might want to "do" the brackets first if they were asked to expand $3(x + 2)$ i.e. they might want to combine the

$x + 2$ into something like $2x$ and then triple the result to get $6x$. Work with area representations should show that $x + 2$ and $2x$ are not equivalent and that $3(x + 2)$ is different from $6x$.

Factorising
Sample Problem 2
The green rectangles below have dimensions x and 1 and have an area of x units squared.
The yellow squares have side lengths of 1 unit and thus an area of 1 unit squared.
How can all the individual areas in $6x + 3$ be arranged into the form of a rectangle?
What are the dimensions of this rectangle?
Rearranging the objects reveals that the dimensions are 3 and $2x + 1$ and ultimately to the realisation that the factors of $6x + 3$ are 3 and $2x + 1$ giving the factorised expression as $3(2x + 1)$.





Notes:

1. Acknowledge $1(6x + 3)$ is a legitimate way of factoring the expression, but as $6x$ and 3 are not mutually prime, further factorisation is possible. In fact it is always the highest common factor of both terms that is used in factoring such expressions.
2. This would be an opportune time for students to practice a range of questions involving expanding and factorising expressions, for example $4x + 8 = 4(x + 2)$ or $2(2x + 4) = 1(4x + 8)$ etc.
3. To solidify the concept of factorising it could be beneficial when, students multiply two expressions, that they are then asked to state the factors of the expression formed.
4. The context of area can be used throughout for many application questions.
5. Questions like simplifying $3(2x + 1) - 2(x + 2)$ and $a(bx + cy + d) + e(fx + gy + h)$ in a purely mathematical context can also be done.

Unit 5: $y = mx + c$ in many representations (The Money Box Activity)

In this Unit students will:

- represent a linear relationship in many ways
- explore the concepts of variables, constants, y-intercept, slope and constant rate of change
- identify the y-intercept and the constant rate of change in many representations
- express relationships in words
- relate rate of change to slope

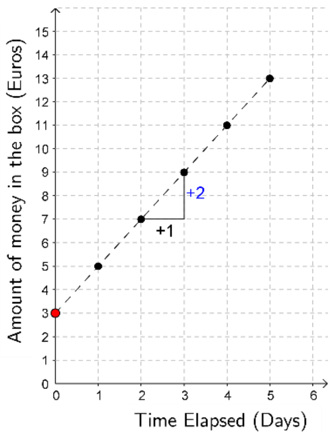
This Unit is based on the Teaching and Learning Plan: Introduction to Patterns and Workshop 4 has a huge amount of information about the example below.

Some of the previous Units mentioned using a table or a graph to represent linear relationships. The majority of work on these types of representation is dealt with in this Unit.

Sample Problem 1

John receives a gift of a money box containing €3 for his birthday. John decides he will save €2 every day, beginning the day immediately following his birthday. Represent this pattern by drawing a table, or diagram.

Time Elapsed (Days)	Amount of money in the box (Euros)	
0	3	
1	5	+2
2	7	+2
3	9	+2
4	11	+2
5	13	+2



Time Elapsed (Days)	Amount of money in the box (Euros)	Amount of money in the box (Euros) (Rewriting to reveal the pattern)
0	3	3+0(2)
1	5	3+1(2)
2	7	3+2(2)
3	9	3+3(2)
4	11	3+4(2)
5	13	3+5(2)
d		3+d(2)

The story or situation given can be represented in table form. The €3 in the story can be seen in the table. The constant increase of €2 per day can be seen in the changes between outputs for consecutive days.

The information in the table can be graphed. Questions like "Where can the starting amount be seen on the graph?" and "How is the constant rate of change evident from the graph?" can be asked.

Students can then be asked to express the relationship (with words) between the numbers of days elapsed and the amount of money in the box i.e. The amount of money in the money box is the initial €3 plus €2 per day.

Students can then be asked to express the relationship (with symbols) between the numbers of days elapsed and the amount of money in the box. $A = 3 + 2d$ where A is the amount of money in the money box in euro and d is the time elapsed in days.

Showing students how the output of 5 can be written as 3+2 and as 3+1(2) and how 7 can be written as 3+2+2 and as 3+2(2) (see table above) can also help students to see how $3 + 2d$ arises.

Attention should be drawn to the €3, in all representations above. The €3 occurs in the story. The €3 is the amount of money when the time elapsed is zero in the table. The €3 is the y-intercept of the graph. €3 is the amount before a number of €2 is added on in the other table. The €3 can also be seen in the relationship expressed in words and the relationship express in symbols.
The constant rate of change, the €2 per day, should be made obvious in all of the representations above. The €2 occurs in the story. The €2 is the change in the outputs between each day in the table. The €2 is the slope of the dotted line in the graph. €2 is the amount added on each day in the other table. The €2 can also be seen in the relationship expressed in words and the relationship express in symbols.

Notes:

1. It is important to note that in this instance the time elapsed is measured in increments of one day. Students will encounter linear relationships with other increments as they progress through their studies.
2. This function in this question does not map the reals to the reals. It maps an arithmetic progression to an arithmetic progression with the ratio of the common difference of the outputs to that of the input being the rate of change.

Work can continue on substitution and solving equations using problems similar to the ones below.

Sample Problem 2 (Substitution)

How much money does John have in his money box 4 days after he got the box?
The answer can be found by (i) analysing the table, (ii) identifying what the output is when the input is the 4^{th} day on the graph, (iii) substituting the 4 into the word expression or (iv) substituting the 4 into the algebraic expression $3 + 2d$ and/or the function $f(d) = 3 + 2d$.
"How much money does John have in his money box 100 days after he got the box?" While the answer can be found by using any of the four methods outlined immediately above, it is unlikely that the table or graphical methods merit consideration as they are inefficient given the size of the numbers involved.

Sample Problem 3 (Solving an Equation)

John wants to buy a new book. The book costs €13. What is the minimum number of days John will have to save so that he has enough money to buy the book?
The answer can be found by (i) analysing the table, (ii) identifying what day has an output of €13 on the graph or (iii) solving $3 + 2d = 13$.

Sample Problem 4 (Solving an Equation)

John wants to buy a new computer game. The game costs €69. What is the minimum number of days John will have to save so that he has enough money to buy the computer game?
Again the answer can be found by using any of the three methods immediately above however it is unlikely that analysing the table should be considered as it is inefficient given the size of the numbers involved.

The Key Features of Functions including the Domain, Range, etc. should be explored and discussed as an integral part this work. Therefore for the function in Sample Problem 1 above for example, the students will be guided to recognise that:

1. The domain is the set of non-negative integers and the range is the set of multiples of two that are greater than or equal four.
2. The function has a y-intercept of 3 and the function does not have an x-intercept.
3. The starting value of three and rate of change of two are constant while the number of days elapsed, d, and the amount of money in the money box, A, are varying.
4. The outputs of the function are always positive and the function is ever-increasing.

5. The average rate of change between outputs when the input is increased by one is constant and is €2 per day.

Notes:

1. It is not feasible to look at the instantaneous rate of change of the function as it does not map the reals to the reals (and also the slope of the tangent to the function is reserved for the senior cycle).
2. Activities from Modular Course 3 that utilised different stimuli could also be used. Some of these activities start with a story, others start with a graph while others start with a table. If students can see the y-intercept (c) and (m) in many representations it should mean we can assess their understanding by seeing if they are able to sketch equations/functions of the form $y = mx + c$ and move from a sketch to $y = mx + c$.
   Questions with negative slopes and fractional slopes should be done too.
3. If it has not occurred naturally already in the questions you use after the "Dots Activity" students should be shown how to go from (1, 5) (2, 8) (3, 11) to (1, 5), (2, 5+3), (3, 5+3+3) and to (1, 5),(2, 5+1(3)),(3, 5+2(3)) and finally to (n, 5+(n–1)(3)).

Inputs	Outputs		
1	5	5	5+0(3)
2	8	5+3	5+1(3)
3	11	5+3+3	5+2(3)
4	14	5+3+3+3	5+3(3)
5	17	5+3+3+3+3	5+4(3)
n			5+(n–1)(3)

Unit 6: Rearranging Linear Formulae

In this Unit students will:

- understand why it is useful to be able to rearrange formulae
- rearrange linear formulae

Repeatedly solving for x when you have an equation in the form $y = mx + c$ is inefficient. The purpose of the Sample Problem below is to show students that there is a need to find a more efficient way of finding inputs when we are given outputs.

Sample Problem

John receives a gift of a money box containing €3 for his birthday. John decides he will save €2 every day, beginning the day immediately following his birthday. After how many days will John have (i) €23 (ii) €45 and (iii) €125 in his money box?

Rearranging the formula $y = 2x + 3$ provides a more efficient approach here.

For $y = 2x + 3$ if we could isolate x then we would have something really useful.
Students should be able to verbalise the steps required to do this i.e. taking 3 from both sides and dividing both sides by 2.

	$y = 2x + 3$	
	$2x + 3 = y$	
−3		−3
	$2x = y - 3$	
÷2		÷2
	$x = \frac{y-3}{2}$	

Rearranging equations of the form $y = mx + c$ to get them in the form $x = \frac{y-c}{m}$ and then substituting is more efficient if we have to solve for many different values of x.
Note: Each time a problem with a linear relationship is given to students from now on the students can be asked to rearrange the equation.

Unit 7: Simplifying the Sum of Like Terms through Adding Functions

In this Unit students will:

- add two functions together to form a new function
- see another reason for finding the sum of expressions $ax + b$ and $cx + d$

The rationale for this Unit is to use functions to see another need for the skill of being able to simplify the sum of like terms. It is sometimes useful to add two functions together to create a new function.

Sample Problem

Mark has a money box; he starts with €6 and adds €2 each day.
Kathy has a money box; she starts with €4 and adds €3 each day.
(i) After how many days will Mark have €40 in his money box? 17 days
(ii) After how many days will Kathy have €40 in her money box? 12 days
(iii) If they combine their money boxes after how many days will they have €40 in their money boxes? 6 days.

The three questions above can be approached in a variety of ways. Tables can be analysed, graphs can be interpreted and equations can be formed and solved. The most efficient way of solving problems like that in question (iii) is to form an equation and use skills of simplifying the sum of like terms (that students know from some engagement with some linear patterns similar to the "Dots Activity"

Time Elapsed (Days)	Money (€)
0	10
1	15
2	20
3	25
4	30
5	35
6	40

$$2x + 6 + 3x + 4 = 40$$
$$2x + 3x + 6 + 4 = 40$$
$$5x + 10 = 40$$
$$-10 \qquad -10$$
$$5x = 30$$
$$\div 5 \qquad \frac{5x}{5} = \frac{30}{5} \qquad \div 5$$
$$x = 6$$

The information given in the question can be viewed as two functions $f(x) = 2x + 6$ and $g(x) = 3x + 4$ and see the steps of the solution as forming a new function $h(x) = f(x) + g(x) = 5x + 10$ and then solving an equation to see what input will give an output of 40.

Unit 8: Comparing Linear Functions, Solving Simultaneous Equations and Inequalities

In this Unit students will:

- compare pairs of linear functions
- decide if two linear relations have a common value
- solve simultaneous equations using a table or a graph
- solve simultaneous equations algebraically (using substitution)
- solve equations of the form $ax + b = cx + d$ for x
- solve inequalities of the form $ax + b < cx + d$ for x

Students will also appreciate the need for an algebraic approach to solving simultaneous equations.

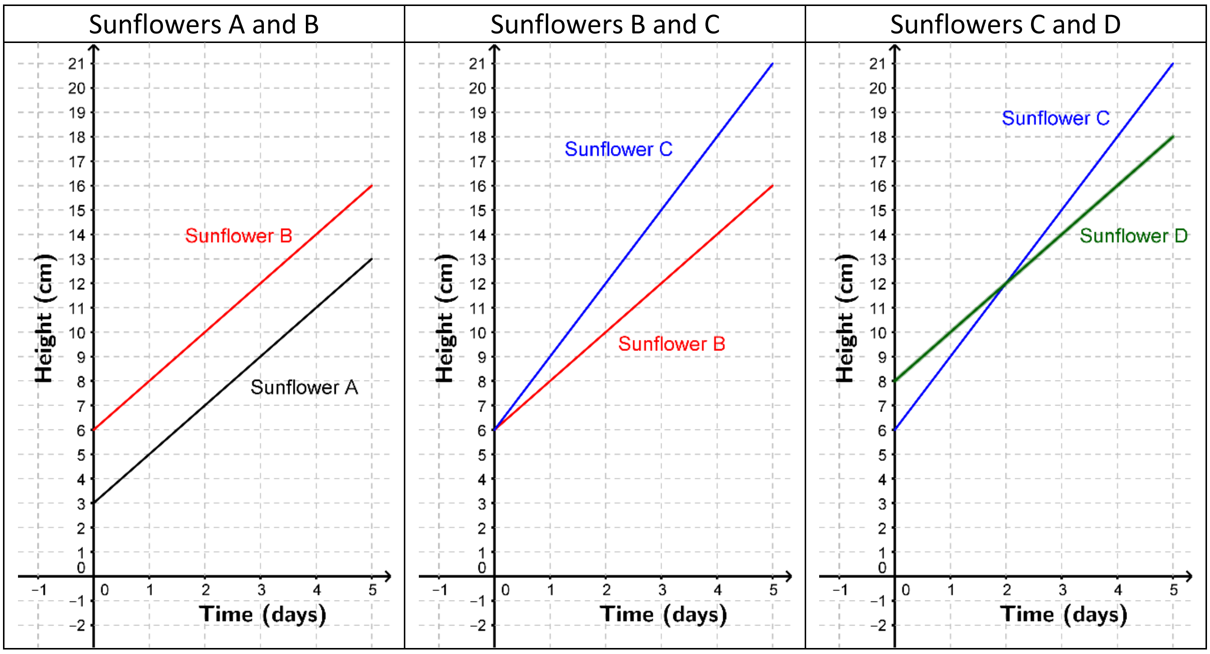
This Unit is based on the sunflowers activity from the Workshop 4 Teacher Resource Booklet.
The activity described the growth pattern of four sunflowers (A, B, C and D) as follows:
Sunflower A: Starting height 3 cm and grows 2 cm per day each day thereafter.
Sunflower B: Starting height 6 cm and grows 2 cm per day each day thereafter.
Sunflower C: Starting height 6 cm and grows 3 cm per day each day thereafter.
Sunflower D: Starting height 8 cm and grows 2 cm per day each day thereafter.
During the workshop some groups compared the growth pattern of A with that of B, others compared B with C while others compared C with D. Their analysis included the use of tables and graphs as shown below

Sunflower A	
Time (Days)	Height (cm)
0	3
1	5
2	7
3	9
4	11
5	13

Sunflower B	
Time (Days)	Height (cm)
0	6
1	8
2	10
3	12
4	14
5	16

Sunflower C	
Time (Days)	Height (cm)
0	6
1	9
2	12
3	15
4	18
5	21

Sunflower D	
Time (Days)	Height (cm)
0	8
1	10
2	12
3	14
4	16
5	18



This activity provides opportunities to investigate a myriad of concepts and skills, including same and different starting values, same and different slopes, parallel lines and intersecting lines. Any concepts and skills from Topics 4.2, 4.3 and 4.4 (below) that can be explored with sunflower type questions should be done here.

Topic	**Description of topic** **Students learn about**	**Learning outcomes** **Students should be able to**
4.2 Representing situations with tables, diagrams and graphs	Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks. Students look at various patterns and make predictions about what comes next.	– use tables, diagrams and graphs as tools for representing and analysing linear, quadratic and exponential patterns and relations (exponential relations limited to doubling and tripling) – develop and use their own generalising strategies and ideas and consider those of others – present and interpret solutions, explaining and justifying methods, inferences and reasoning
4.3 Finding formulae	Ways to express a general relationship arising from a pattern or context.	– find the underlying formula written in words from which the data are derived (linear relations) **– find the underlying formula algebraically from which the data are derived (linear, quadratic relations)**
4.4 Examining algebraic relationships	Features of a relationship and how these features appear in the different representations. Constant rate of change: linear relationships. Non-constant rate of change: quadratic relationships. Proportional relationships.	– show that relations have features that can be represented in a variety of ways – distinguish those features that are especially useful to identify and point out how those features appear in different representations: in tables, graphs, physical models, and formulas expressed in words, and algebraically – use the representations to reason about the situation from which the relationship is derived and communicate their thinking to others – recognise that a distinguishing feature of quadratic relations is the way change varies – discuss rate of change and the y-intercept; consider how these relate to the context from which the relationship is derived, and identify how they can appear in a table, in a graph and in a formula – decide if two linear relations have a common value – investigate relations of the form y=mx and y=mx +c – recognise problems involving direct proportion and identify the necessary information to solve them

Notes:

1. If students have not done so already they should be become adept at moving easily between the graph of a function of the form $y = mx + c$ and its algebraic form.
2. If students are sketching the graphs of various functions a useful one for later would be the sketch of an artificial sunflower. This would have the form $f(x) = a$, where $a, x \in R^{+} \cup \{0\}$. It will be important to understand how to draw functions of type $f(x) = a$ for understanding solving quadratics of the form $x^2 + 5x + 6 = 42$, which will be looked at from the point of view of comparing $f(x) = x^2 + 5x + 6$ with $g(x) = 42$.

Solving Simultaneous Equations Graphically and from a Table

Sample Problem

When will sunflowers C and D have the same height?

The third diagram above, which compared sunflowers C and D, should be discussed in detail as an introduction to solving simultaneous equations. Students should be asked to identify when sunflowers C and D have the same height i.e. the same height at the same time. They should be able to see the answer that both sunflowers have the same height, 12 cm, 2 days after the start by analysing the graph and/or the table. Topic 5.2 has the learning outcome “use graphical methods to find approximate solutions where $f(x) = g(x)$.”

Solving Simultaneous Equations using Substitution

C: $y = 3x + 6$

D: $y = 2x + 8$

Substitution can be used here to solve this system of equations i.e. if $y = 3x + 6$, then $3x + 6$ can be substituted for y in $y = 2x + 8$ to get $3x + 6 = 2x + 8$.

$3x + 6 = 2x + 8$
$x + 6 = 8$
$x = 2$
$y = 2x + 8$
$y = 2(2) + 8$
$y = 12$ The sunflowers have the same height, 12 cm, after 2 days.

Notes:

1. Substitution is easy to use here because both equations are in the form $y = mx + c$ and m is a whole number in both equations.
2. The method will be used later to solve, for example at Leaving Certificate Ordinary Level "one linear equation and one equation of order 2 with two unknowns (restricted to the case where either the coefficient of x or the coefficient of y is ± 1 in the linear equation) and interpret the results".
3. It could be difficult for a student to rearrange equations C and D into the form $ax + by = c$ to make them suitable for the elimination method. The equations would look like: $3x - y = -6$ and $2x - y = -8$.
4. It would be useful to show students that the problem could be represented by $f(x) = 3x + 6$ and $g(x) = 2x + 8$ and the method above can be used to find when $f(x) = g(x)$.
5. Comparing $f(x) = ax + b$ with $g(x) = c$; $a, b, c \in Z; x \in R$ and $a \neq 0$, $b \neq 0$, $c \neq 0$ could also be useful for understanding that the 27 in $4x + 7 = 27$ can be seen as not just a constant or just the y-value of the point (5, 27). The 27 can also be interpreted as the function $g(x) = 27$ which has an output of 27 irrespective of the input and so is represented in graphical form by the horizontal line $y = 27$. Viewing questions like this could be beneficial to students.
6. Students have solved equations of the form $ax + b = c$ earlier. They would have the ability at that stage to solve equations of the form $ax + b = cx + d$ but it might be best to wait until now to practice the skill of solving equations of the form $ax + b = cx + d$. This is because students can now see a need for the skill.

At this stage, students have seen three ways of solving simultaneous equations; using a table, a graph or algebraic methods. It is important that students understand the advantages of the algebraic method. The table takes time to create if the x-values required are very large. Additionally, it is hard to estimate the solution precisely when the one or both of the co-ordinates of the point of intersection are not whole numbers. To see an example of this type of material click here.

Inequalities

Having determined when two sunflowers have the same height it is inevitable that when one is taller or shorter than the other should also be considered. This, is turn, facilitates introducing the use of the symbols relating to equations and inequalities. Many of the questions thus encountered can be solved graphically before an algebraic method is introduced.

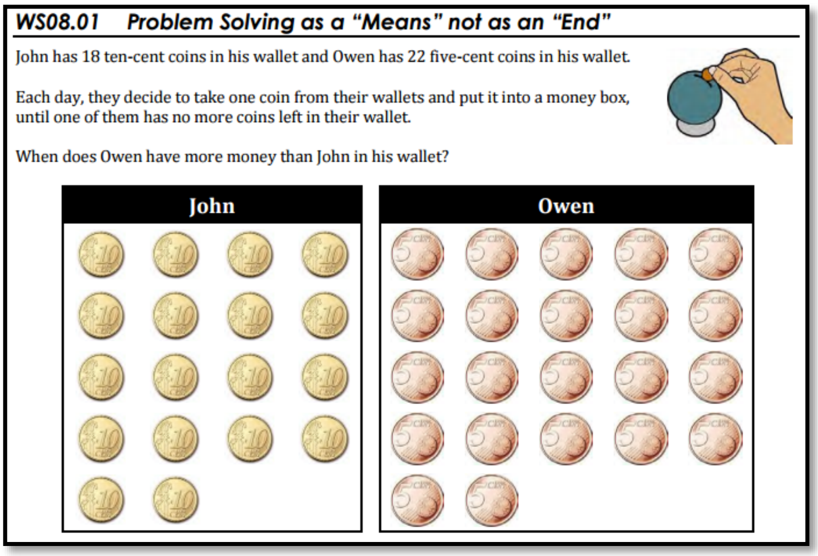
Sample Problem

When will sunflower C be taller than sunflower D?

$3x + 6$ is an expression for the height of sunflower C at any time, x.
$2x + 8$ is an expression for the height of sunflower D at any time, x.
To work out when sunflowers C and D had the same height the equation $3x + 6 = 2x + 8$ was used.
To work out when sunflower C will be taller than sunflower D the inequality $3x + 6 > 2x + 8$ can be used.

Notes:

1. This problem is related to topic 5.2 of the syllabus which has the learning outcome "interpret inequalities of the form $f(x) \leq g(x)$ as a comparison of functions of the above form; use graphical methods to find approximate solution sets of these inequalities and interpret the results".
2. When answering this type of question it is useful to keep this other learning outcome from Topic 5.2 in mind "graph solution sets on the number line for linear inequalities in one variable" as this is an opportune time to practice the skill of graphing solution sets on number lines because students can see a need for the skill.
3. The two wallets question from Workshop 8 is a good question to use in addition to or instead of the sunflowers to introduce algebraic inequalities.



Optional Work

Topic 4.7 has the learning outcome solve linear inequalities in one variable of the form
$g(x) \leq k$ where $g(x) = ax + b, a \in \mathbb{N}$ and $b, k \in \mathbb{Z}$;
$k \leq g(x) \leq h$ where $g(x) = ax + b$, and $k, a, b, h \in \mathbb{Z}$ and $x \in \mathbb{R}$.
Inequalities of the form $k \leq g(x) \leq h$ where $g(x) = ax + b$, and $k, a, b, h \in \mathbb{Z}$ and $x \in \mathbb{R}$ could be dealt with **now or later**.

Sample Problem

When will Sunflower A have a height between 17 cm and 29 cm inclusive?
The problem can be solved by analysing a table, or interpreting a graph.
Using algebraic skills the problem becomes find the solution set of $17 \leq 2x + 3 \leq 29$.
Three methods are shown below:

$$\begin{gathered} 17 \leq 2x+3 \leq 29 \\ 17-3 \leq 2x+3-3 \leq 29-3 \\ 14 \leq 2x \leq 26 \\ \frac{14}{2} \leq \frac{2x}{2} \leq \frac{26}{2} \\ 7 \leq x \leq 13 \end{gathered}$$

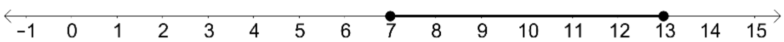
$$\begin{gathered} 17 \leq 2x+3 \leq 29 \\ \underline{-3 \qquad -3 \qquad -3} \\ \frac{14}{2} \leq \frac{2x}{2} \leq \frac{26}{2} \\ 7 \leq x \leq 13 \end{gathered}$$

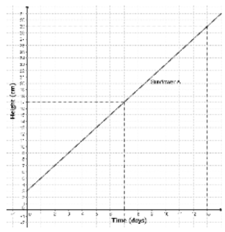
$$17 \leq 2x+3 \leq 29$$

$$\begin{gathered} 17 \leq 2x+3 \\ 17-3 \leq 2x+3-3 \\ 14 \leq 2x \\ \frac{14}{2} \leq \frac{2x}{2} \\ 7 \leq x \end{gathered} \qquad \begin{gathered} 2x+3 \leq 29 \\ 2x+3-3 \leq 29-3 \\ 2x \leq 26 \\ \frac{2x}{2} \leq \frac{26}{2} \\ x \leq 13 \end{gathered}$$

$$7 \leq x \leq 13$$

Sunflower A will have height between 17 cm and 29 cm inclusive from the start of the 7^{th} day up to and including the beginning of the 13^{th} day.





Unit 9: Elimination Method for Solving Simultaneous Equations

In this Unit students will:

- solve simultaneous equations using the elimination method

Students will also see the need for an algebraic approach to solving simultaneous equations.

This topic could be introduced by giving students a problem to solve a couple of days before they are introduced to the elimination method in class.

Sample Problem

There are 25 toys in a playground. The toys are either bicycles (2 wheels) or tricycles (3 wheels). In total in the playground there are 61 wheels. How many bicycles are in the playground?

The expected approach here is that students would use trial and improvement.

10(2)+15(3)	15(2)+10(3)	14(2)+11(3)
= 20+45	= 30+30	= 28+33
=65	= 60	=61

A more sophisticated solution would use the fact that an odd number of tricycles are needed. This would reduce the work slightly for those that spot it.

A similar puzzle can be given with much larger numbers could be shown to students to show why it would be a good idea to have a more efficient method than trial and improvement.

To build towards the elimination method the equations can be formed using words:

The number of bicycles + the number of tricycles = The number of toys

Two times the number of bicycles + Three times the number of tricycles = The number of wheels

Then letters can be used, with each letter defined clearly

$b + t = 25$	b = The number of bicycles (i.e. not b = bicycles)
$2b + 3t = 61$	t = The number of tricycles (i.e. not t = tricycles)

This could be solved algebraically now or some work could be done on the mechanics of simultaneous equations and return to solve the equations when they are able to do the method.

Two More Ideas for the Elimination Method

Idea 1

It might be a good idea to show students some equations that have the same coefficient for x (or y) so students see what they would like to have.

Sample Problem

Can you work out what value x or y has in the following sets of equations?

$5x + 2y = 35$ $\qquad$ $x + 3y = 24$
$5x + 8y = 95$ $\qquad$ $6x + 3y = 39$

In the first set of equations the coefficient of x is the same in both equations and as the coefficient of y is increased on the left hand side of the equation by 6 the total on the right is increased by 60. So $y = 10$. From there it is easy to work out the value of x, which is 3. A similar strategy can be used for the second set of equations.

If students can understand how to solve the systems of equations above then the strategy for future questions can be "What do I need to do to the system of equations I have to make them look like the systems of equations I am able to solve?". Students can then implement this strategy for

$b + t = 25$
$2b + 3t = 61$

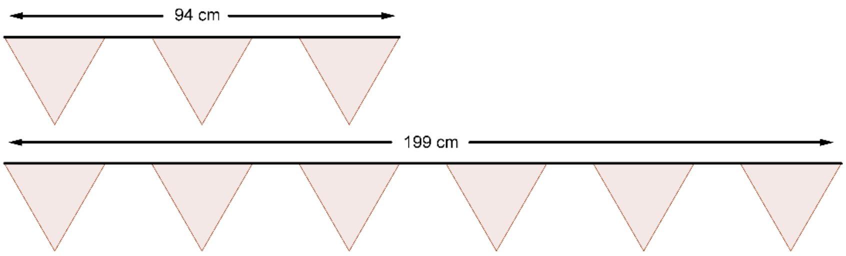
which is the question at the start of the Unit.

Idea 2

A visual approach could be used. The following question could be used.

Sample Problem

"Lisa is putting up flags for a party. The flags are all the same size and are spaced equally along the string. What is the width of the base of each flag? What is the length of the space between each flag?"



The width of the base of three flags + the width of two gaps = 94 cm $\qquad 3x + 2y = 94$
The width of the base of six flags + the width of five gaps = 199 cm $\qquad 6x + 5y = 199$

If the first pattern is repeated we get
The width of the base of six flags + the width of four gaps = 188 cm $\qquad 6x + 4y = 188$
The difference between 199 cm and 188 cm is 11 cm. The width of a gap is 11 cm. $\qquad y = 11$
From there we can work out that the width of the base of a flag is 24 cm $\qquad x = 24$

The width of the base of a flag is 24 cm.
The length of the space between each flag is 11 cm

This question should show that it is useful that the coefficient of x (or y) is the same in both equations. Another alternative would be that the coefficients of x (or y) would have the same magnitude but one coefficient would have a positive value and the other would be negative.

Unit 10: Algebraic Expressions and Equations that have Fractions in them

In this Unit students will:

- form algebraic expressions that have fractions in them
- form algebraic equations having fractional coefficients
- solve algebraic equations having fractional coefficients

Sample Problem

Mary starts with €3 in her money box and puts €2 in each week.
John starts with €2 in his money box and puts €2 in each week.
They both want to save to buy a computer game that costs €37.
Mary says she will give half of the money in her money box.
John says he will give a quarter of the money in his money box.
After how many weeks will they have enough money for the game?

x is the number of weeks elapsed.
The amount of money in Mary's Money Box after x weeks is $2x$+3 euro.
The amount of money in John's money box after x weeks is $3x$+2 euro.
The amount of money Mary is willing to invest is $\frac{2x+3}{2}$ euro
The amount of money John is willing to invest is $\frac{3x+2}{4}$ euro
The combined amount they are willing to invest is $\frac{2x+3}{2} + \frac{3x+2}{4}$
The equation to solve is $\frac{2x+3}{2} + \frac{3x+2}{4} = 37$
They will have the €37 they need for the game 20 weeks after they start saving.
The same problem can be used to simplify the sum of the algebraic expressions to get $\frac{7x+8}{4}$.
Additional questions include how long it would take Mary on her own and John on his own.

Appendix 1: Relations Without Formulae

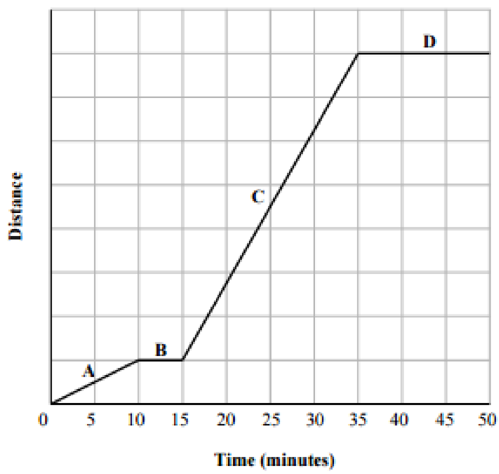
Relations without formula is topic 4.5 of the Junior Certificate Syllabus. It has not been placed in the sequence of topics in this document. This topic can be addressed at any stage and/or throughout. When students engage with this Unit they will:

- use graphs to represent phenomena quantitatively
- explore graphs of motion
- make sense of quantitative graphs and draw conclusions from them
- make connections between the shape of a graph and the story of a phenomenon
- describe both quantity and change of quantity on a graph

Given that every function is a relation, in teaching this Unit the Key Features of Functions should always be kept in mind.
The key features of functions are:
1. The domain and range
2. Where the graph of the function meets the axes.
3. What is constant and what varies in the function?
4. The behaviour of the graph of the function
5. The rate of change of the function

The language of comparison of functions from Unit 8 can also be used for these relations, for example, less than, greater than.
Motion sensors are a very good way of helping students understand relations without formula and/or functions.
The Workshop 4 Teacher Resource Booklet and this resource contain some ideas for teaching relations without formula.
The concept of rate of change, that was dealt with in the Money Box Activity in Unit 5, could be further explored with questions similar to the one below.

Gráinne is taking part in a training session.
The graph shows the distance she travelled during the session.
The four parts of the graph are labelled **A**, **B**, **C**, and **D**.



(a) Write the letters **A**, **B**, **C**, and **D** into the table to match each description with the correct part of the graph.

Description	Part of the Graph
Gráinne runs for 20 minutes	
Gráinne stops for 15 minutes	
Gráinne walks for 10 minutes	
Gráinne stops for 5 minutes	

Appendix 2: Repeating Patterns

Generating arithmetic expressions from repeating patterns is topic 4.1 of the Junior Certificate Syllabus. It has not been placed into this sequence of topics. This document places the "Dots Activity" from the Seminar 2014–2015 as a key task for formally introducing the concept of a variable to students.

When students engage with this Unit they will:

- use tables to represent a repeating-pattern situation
- generalise and explain patterns and relationships in words and numbers
- write arithmetic expressions for particular terms in a sequence

Student Activities 1A and 1B of Teaching and Learning Plan: Introduction to Patterns are similar to the "Dots Activity" in that the students have part of a pattern, for example unifix cubes with the pattern Red-Black-Red-Black etc. and they are asked questions similar to the ones from the "Dots Activity".

Multiplies of two are important in Red-Black-Red-Black unifix cubes activity. Expressions such as $2n$, $2n - 1$ can be formed.

Multiplies of three are important in Yellow-Black-Green unifix cubes activity. Expressions such as $3n$, $3n - 1$, $3n - 2$ can be formed.

Repeating patterns could be explored when studying Natural Numbers to better understand multiples.

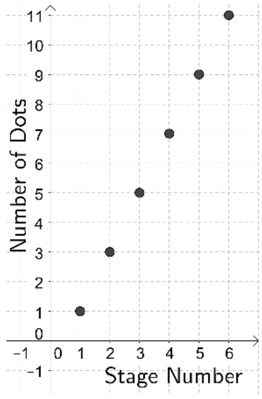
Appendix 3: Key Questions

Each Unit contains a number of questions. Some of these are key questions that should be returned to in order that new knowledge, skills and concepts can be applied in contexts with which they are familiar. Key questions include the "Dots Activity" in Unit 1, the Money Box Activity in Unit 5 and the Sunflowers Activity in Unit 8. To illustrate this, examples of how the "Dots Activity" can be revisited are illustrated below.

Generalising, Substitution and Multiple Representations – Unit 1

In Unit 1 students use the "Dots Activity" to (i) express a relationship in words, (ii) develop and use their own generalising strategies and ideas and consider those developed by others and (iii) make use of letters to represent variables. There are also opportunities to practice the concept and skill of substitution; for example, if students recognise that the number of dots in each stage is "Twice(s)–1", where s is the stage number and need to determine the number of dots in stage 100.

Stage Number	Number of Dots
1	1
2	3
3	5
4	7
5	9
6	11



Solving Equations – Unit 2

At the end of Unit 1 students can be challenged with the following problem: "Which stage has 47 dots?". The problem can be solved in many ways. The skill of solving an equation using formal algebraic methods is addressed in Unit 2. Once the skill is learned, students can return to the problem and find the stage number by solving the linear equation $2s - 1 = 47$.

$y = mx + c$ in many representations (The Money Box Activity) – Unit 5

Once the "Money Box" activity is addressed the constant rate of change and the y-intercept can be analysed in greater depth.

Rearranging Linear Formulae – Unit 6

$t = 2s - 1$ can be rearranged to become $s = \frac{t+1}{2}$ which is useful for working out the stage number when the number of dots is known.

Comparing Linear Functions, Simultaneous Equations and Inequalities – Unit 8

Solving the equation $2s - 1 = 47$ can be viewed as comparing the functions $f(s) = 2s - 1$ and $g(s) = 47$.

Two dot patterns could be explored to see when both have the same number of dots in each stage.

$2s - 1 > 47$ can be viewed as the comparison of the functions $f(s) = 2s - 1$ and $g(s) = 47$.

Two dot patterns could be explored to see for what stages one pattern has a greater number of dots than the other pattern.

Key Features of Functions

The key features of functions are:
1. The domain and range
2. Where the graph of the function meets the axes.
3. What is constant and what varies in the function?
4. The behaviour of the graph of the function
5. The rate of change of the function

For the "Dots Activity":

1. The domain is the set of natural numbers and the range is the set of odd numbers greater than or equal to one.
2. The function $f(s) = 2s - 1, s \in \mathbb{N}$ does not have a y-intercept or an x-intercept.
3. If the function is expressed as $f(s) = 1 + (s - 1)2, s \in \mathbb{N}$ the starting value of one and rate of change of two are constant while the stage number, s, and number of dots in each stage, $f(s)$, are varying.
4. The outputs of the function are always positive and the function is ever increasing.
5. The average rate of change between outputs when the input is increased by one is constant and is 2 dots per stage.