Patterns
A Relations Approach to Algebra

Junior Certificate Syllabus
Leaving Certificate Syllabus

Project Maths
Tionscadal Mata
Development Team

Contents

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Chapter 1: Relationship to syllabuses, introduction and unit learning outcomes

Junior Certificate Mathematics: Draft Syllabus

Topic	Description of topic *Students learn about*	Learning outcomes *Students should be able to*
4.1 Generating arithmetic expressions from repeating patterns	Patterns and the rules that govern them; students construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a correspondence from each input to each output.	• use tables to represent a repeating-pattern situation • generalise and explain patterns and relationships in words and numbers • write arithmetic expressions for particular terms in a sequence
4.2 Representing situations with tables, diagrams and graphs	Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks. Students look at various patterns and make predictions about what comes next.	• use tables, diagrams and graphs as tools for representing and analysing linear, quadratic and exponential patterns and relations (exponential relations limited to doubling and tripling) • develop and use their own generalising strategies and ideas and consider those of others • present and interpret solutions, explaining and justifying methods, inferences and reasoning
4.3 Finding formulae	Ways to express a general relationship arising from a pattern or context.	• find the underlying formula written in words from which the data is derived (linear relations) • find the underlying formula algebraically from which the data is derived (linear, quadratic relations)
4.4 Examining algebraic relationships	Features of a relationship and how these features appear in the different representations. Constant rate of change: linear relationships. Non-constant rate of change: quadratic relationships. Proportional relationships.	• show that relations have features that can be represented in a variety of ways • distinguish those features that are especially useful to identify and point out how those features appear in different representations: in tables, graphs, physical models, and formulas expressed in words, and algebraically • use the representations to reason about the situation from which the relationship is derived and communicate their thinking to others • recognise that a distinguishing feature of quadratic relations is the way change varies • discuss rate of change and the y-intercept, consider how these relate to the context from which the relationship is derived, and identify how they can appear in a table, in a graph and in a formula • decide if two linear relations have a common value (decide if two lines intersect and where the intersection occurs) • investigate relations of the form $y = mx$ and $y = mx + c$ • recognise problems involving direct proportion and identify the necessary information to solve them

Leaving Certificate Mathematics: Draft Syllabus

Students learn about	Students working at FL should be able to	In addition, students working at OL should be able to	In addition, students working at HL should be able to

3.1 Number systems	• recognise irrational numbers and appreciate that **R**≠**Q**
	• revisit the operations of addition, multiplication, subtraction and division in the following domains:
	  • **N** of natural numbers
	  • **Z** of integers
	  • **Q** of rational numbers
	  • **R** of real numbers
	and represent these numbers on a number line
	• appreciate that processes can generate sequences of numbers or objects
	• investigate patterns among these sequences
	• use patterns to continue the sequence
	• generate rules/formulae from those patterns develop decimals as special equivalent fractions strengthening the connection between these numbers and fraction and place value understanding
	• consolidate their understanding of factors, multiples, prime numbers in **N**
	• express numbers in terms of their prime factors
	• appreciate the order of operations, including brackets
	• express non-zero positive rational numbers in the form a x10^n, where $n \in$ **N** and $1 \leq a < 10$ and perform arithmetic operations on numbers in this form
		• generalise and explain patterns and relationships in algebraic form
		• recognise whether a sequence is arithmetic, geometric or neither
		• find the sum to n terms of an arithmetic series
			• verify and justify formulae from number patterns
			• investigate geometric sequences and series
			• prove by induction solve problems involving finite and infinite geometric series including applications such as recurring decimals and financial applications, e.g. deriving the formula for a mortgage repayment
			• derive the formula for the sum to infinity of geometric series by considering the limit of a sequence of partial sums

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Strand 4: Algebra

This strand builds on the relations-based approach of Junior Certificate where the five main objectives were

1. to make sense of letter symbols for numeric quantities
2. to emphasise relationship based algebra
3. to connect graphical and symbolic representations of algebraic concepts
4. to use real life problems as vehicles to motivate the use of algebra and algebraic thinking
5. to use appropriate graphing technologies (graphing calculators, computer software) throughout the strand activities.

Students build on their proficiency in moving among equations, tables and graphs and become more adept at solving real world problems.

Junior Certificate Draft Syllabus page 26,
Leaving Certificate Draft Syllabus page 28.
National Council for Curriculum and Assessment (2011), Mathematics Syllabus, Junior Certificate [online] available: http://www.ncca.ie/en/Curriculum_and_Assessment/Post-Primary_Education/Project_Maths/Syllabuses_and_Assessment/Junior_Cert_Maths_syllabus_for_examination_in_2014.pdf

National Council for Curriculum and Assessment (2011), Mathematics Syllabus, Leaving Certificate [online] available: http://www.ncca.ie/en/Curriculum_and_Assessment/Post-Primary_Education/Project_Maths/Syllabuses_and_Assessment/Leaving_Cert_Maths_syllabus_for_examination_in_2013.pdf

Accessed September 2011.

A functions based approach to algebra

The activities below focus on the important role that functions play in algebra and characterise the opinion that algebraic thinking is the capacity to represent quantitative situations so that relations among variables become apparent. Emphasising common characteristics helps us to think of functions as objects of study in and of themselves and not just as rules that transform inputs into outputs. Students examine functions derived from some kind of context e.g. familiar everyday situations, imaginary contexts or arrangements of tiles or blocks. The functions-based approach enables students to have a deep understanding in which they can easily manoeuvre between equations, graphs and tables. It promotes inquiry and builds on the learner’s prior knowledge of mathematical ideas.

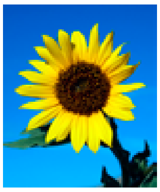
In this unit:

1. Students link between words, tables, graphs and formulae when describing a function given in a real life context.
2. Students come to understand functions as a set of inputs and a set of outputs and a correspondence from each input to each output.
3. Students identify what is varying and what is constant for a function.
4. Students will understand the terms variable and constant, dependent and independent variable and be able to identify them in real life contexts.
5. Students will build functional rules from recursive formulae.
6. Students focus on linear functions first, identifying a constant rate of change as that which characterises a linear function. Students will identify this as constant change in the y - values for consecutive x - values. They will identify this change in y per unit change in x as the slope of the straight line graph.
7. Students will see that linear functions can have different 'starting values' i.e. value of y when $x = 0$. They will identify this as the y – intercept.
8. Students will see how the slope and y intercept relate to the context from which the function is derived.
9. Students will explore a variety of features of a function and examine how these features appear in the different representations–
    a. Is the function increasing, decreasing, or staying the same? If the function is increasing the slope is positive, if decreasing the slope is negative and if constant it has zero slope.
    b. Is the function increasing/decreasing at a steady rate or is the rate of change varying? If it is increasing at a constant rate this is characteristic of a linear function.
10. Students will investigate real life contexts which lead to quadratic functions where the rate of change is not constant but the rate of change of the rate of change is constant. Students will identify this as constant change of the changes in the y values for consecutive x values
11. Students will examine instances of proportional and non proportional relationships for linear functions. Students will know the characteristics of a proportional relationship when it is graphed – it is linear and passes through the origin (no 'start-up' value, i.e. no y - intercept) and the 'doubling strategy '[if x is doubled (or increased by any multiple) then y is doubled (or increased by the same multiple)].

12. Students investigate contexts which give rise to quadratic functions through the use of words, tables, graphs and formulae.
13. When dealing with graphs of quadratic functions students should contrast the quadratic with the linear functions as follows:
    - the graphs are non linear
    - the graphs are curved
    - the changes are not constant
    - the change of the changes is constant
    - the highest power of the independent variable is 2
14. Students should investigate situations involving exponential functions using words, tables, graphs and formulae and understand that exponential functions are expressed in constant ratios between successive outputs.
15. Students should see applications of exponential functions in their everyday lives and appreciate the rapid rate of growth or decay shown in exponential functions.
16. Students should discover that for cubic functions the 'third changes' are constant.
17. Students will understand that x^3 increases faster than x^2 and the implications of this in nature and in design.
18. Students will investigate situations where inverse proportion applies and contrast these with linear, quadratic, and exponential relationships. Students will see that the product of the variables is a constant in these situations.
19. Students will interpret graphs in cases of 'relations without formulae'.

Chapter 2: Comparing Linear Functions

Student Activity – comparing linear functions



(This activity may be introduced with students at any stage - JCOL, LCFL, LCOL) GRAPH PAPER REQUIRED

The activities give students an opportunity to link this strand with co-ordinate geometry and can be taken before or after it. Through the study of patterns of change, they can get a more meaningful understanding for such characteristics as slopes, intercepts, equation of a line, etc.

In this activity, students

- Identify patterns and describe different situations using tables and graphs, words and formulae
- Identify independent and dependent variables and constants
- Identify 'start amount' in the table and formula and as the y-intercept of the graph
- Identify the rate of change of the dependent variable in the table, graph (as the slope) and formula
- Identify linear relationships as having a constant change between successive y values (outputs)
- Know that parallel lines have the same slope (same rate of change of y with respect to x)
- Connect increasing functions with positive slope, decreasing functions with negative slope and constant functions with slope of zero. (It is important that students realise that tables, graphs and algebraic formulae are just different ways to represent a situation.)

The following data shows the measured heights of 4 different sunflowers on a particular day and the amount they grew in centimetres each day afterwards.

Sunflower	a	b	c	d
Start height/cm	3	6	6	8
Growth per day/cm	2	2	3	2

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Investigate how the height of each sunflower changes over time. Is there a pattern to the growth? If you find a pattern can you use it to predict how tall the sunflowers will be in say 30 days, or in any number of days? What different representations could you use to help your investigation?

Represent this information in a table similar to the one below, showing the height of the sunflowers over a six day period for each situation (a), (b), (c) and (d).

Time in days	Height in cm	Change
0		

1. Where is the 'start height' for each plant seen in the tables?
2. Where is the amount the plant grows by each day seen in the tables?
3. What do you notice about the change between successive outputs for each the tables?
   (4 and 5 may be omitted as the main aim of the activity is for students to recognise the features of a linear relationship in a table and in a graph.)
4. For each of the situations and tables (a), (b), (c), and (d) identify 2 values which stay the same and 2 values which vary.

Situation and Table	Varying	Staying the same
a		
b		
c		
d		

We call the values which vary 'variables' and the values which stay the same 'constants'.

We have identified 2 variables. Which variable depends on which?

We call one variable the dependent variable and we call the other variable the independent variable.

5. Which is the independent variable and which is the dependent variable in each case above?

Plotting graphs of the situations represented in table 1:

Using graph paper, on the first set of axes, graph situations a and b.

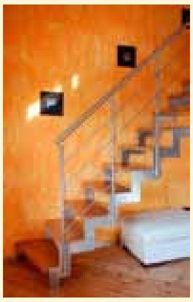
Which variable do you think should go on which axis? Discuss.

We usually plot the independent variable on the x-axis and the dependent variable on the y-axis.

We can think of the x values as the inputs and the y values as the outputs.

1. What observations can you make about graphs of a and b?
2. How is each observation seen in the situation?
3. How is each observation seen in the table?
4. Are the sunflowers a and b ever the same height? Explain.

y - intercept and slope



5. Where are the start heights seen in the graphs?
   We call these values the y - intercepts of the graphs.
6. Where are the growth rates seen in the graphs?
   (As the day increases by 1 what happens to the height?)
   We call this value the slope of the graph – the change in y per unit change in x.
7. If the number of days increases by 3, what does the height increase by?
   What is the average increase in height per day?
   Is this the same as the slope?
8. Can you come up with a way of calculating slope?
9. Going back to the sunflower example, what does the slope correspond to?

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Formulae in words and symbols: The following can be applied to Sunflowers (a) and (b) JC Syllabus 4.3 – HL in bold

4.3 Finding Formulae	Ways to express a general relationship arising from a pattern or context	• finding the underlying formula written in words from which the data is derived (linear relations) • find the underlying formula algebraically from which the data is derived (linear, quadratic relations)

Using the table and the graph (which are just different representations of the same situation):

1. Describe in words the height of the plant in (a) on a particular day (e.g. day 4) in terms of its 'start height' and its growth rate per day.
2. Describe in words the height h of the plant in (a) on any day d, in terms of its 'start height' and the amount it grows by each day.
3. Write a formula for the height h of the plant in (a) on any day d, in terms of its 'start height' and the amount it grows by each day.
4. Identify the variables (dependent and independent) and the constants in the formula.

Formulae in words and symbols: Sunflower (b)

5. Where is the 'start height' seen in the formulae for a and b? Where is the amount grown each day seen in the formula for a and b?
6. Identify where each variable and constant in the formulae appears in the graphs.

Possible Homework or class group work: Investigate the pattern of sunflower growth in situations b and c using tables, graphs and formulae as you did for situations a and b.

Next class: Investigate the pattern of sunflower growth in situations c and d using tables, graphs and formulae as you did for situations a and b and b and c.

Intersecting graphs c and d

1. After how many days are the sunflowers in situations (c) and (d) the same height?
2. Why do you think sunflower (c) overtakes sunflower (d) even though sunflower (d) starts out with a greater 'start height' than sunflower (c)?

Increasing functions and positive slopes

1. As the number of days increased what happened to the height of the sunflower in each situation?
2. What sign has the slope of the graph in each situation?

Line with constant slope

Monica decided to plant a plastic sunflower whose height was 30cm. Represent this information in a table, a graph and an algebraic formula for day 0 to day 5.

1. What shape is the graph?
2. Contrast the situation with the previous sunflower situations. Why is the graph this shape?
3. Where is the 30cm in the table? Where is the 30cm in the graph?
4. How much does the sunflower grow each day? Where is that in the table? Where is it in the graph?
5. As the number of days increased what happened to the height of the sunflower?
6. What is the slope of the graph?
7. What formula would describe the height of the sunflower?
8. Where is the 30cm in the formula?
9. What are the limitations to the model presented for sunflower growth?
   It is important for students to realise early on that it is impossible to mirror accurately many real life situations with a model and they must always recognise the limitations of whatever model they use.

Line with a negative slope:

You have €40 in your money box on Sunday. You spend €5 on your lunch each day for 5 consecutive days (take Monday as day 1).

As the variable on the x - axis increases what happens to the variable on the y - axis?

Is sunflower growth a realistic situation for negative slope?

Can you think of other real life situations which would give rise to linear graphs with negative slopes?

Summary of this Student Activity:

The height of each plant depends on how old it is. We have an input (time in days) and an output (height of the plant) which depends on the input. We say that the output (in this case height of the plant) is the dependent variable and the input (time) is the independent variable. When we use x and y, x is the independent variable and y is the dependent variable.

We see that a, b and d have different starting heights and b and c have the same starting heights. The starting height appears in the table as the height on day 0 and in the graph as the y value when the x value (number of days) is zero. In the formula it appears on its own as a constant.

We see that there is a constant rate of change each day for the plant's height in each situation. In the table it appears as a constant change for successive y outputs which characterises linear relationships. In the graph it appears as the slope of the line – constant change in y for each unit change in x. In the formula for a particular situation this constant rate of change is multiplied by the independent variable, in this case the number of days. When two plants have the same rate of change in their growth per day their graphs are parallel, i.e. they have the same slope.

When two plants have different growth rates per day the graphs will intersect and the plant with the higher growth rate will eventually overtake the plant with the lower growth rate even if the one with lower growth rate starts at a greater height.

Student Activity – draw a graph from the situation without a table of values JCOL

As in the previous activities, there are obvious links to co-ordinate geometry and to functions/graphs.

It is important that students are confident at sketching/drawing linear graphs without the need to formally set out a table, plot points and then draw the graph.

Eventually, they should be able to use this knowledge/skill when presented with a similar set of conditions.

(Discussion point: Is the linear relationship a continuous one or just discrete values? Is joining the points always appropriate?)

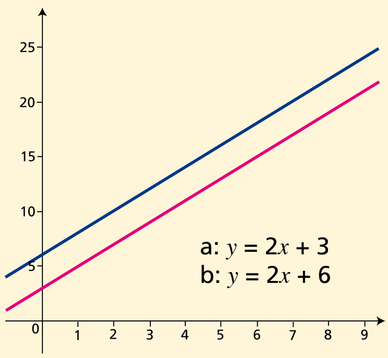
Draw graphs of linear functions without making a table of values, by comparing them to the graphs you have already made.

1. On the same axes you used for **Student Activity 1** (situations a and b) sketch a graph of the following sunflower growth pattern: starting height 5cm, grows 2cm each day.
   Compare this graph to the graphs of situations a and b? Explain the comparison.

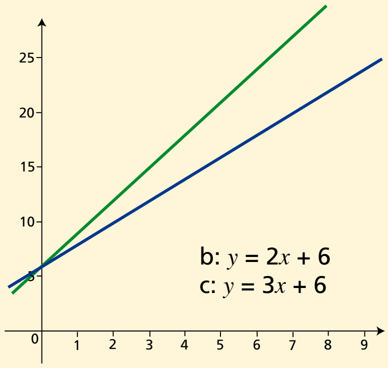
2. On the same axes you used for **Student Activity 1** (situations b and c) sketch a graph of the following sunflower growth pattern: starting height 6cm, grows 4cm each day.
   Compare this graph to the graphs of situations b and c? Explain the comparison.

3. On the same axes you used for **Student Activity 1** (situations c and d) sketch a graph of the following sunflower growth pattern: starting height 6cm, grows 2cm each day
   Compare this graph to the graphs of situations c and d? Explain the comparison.

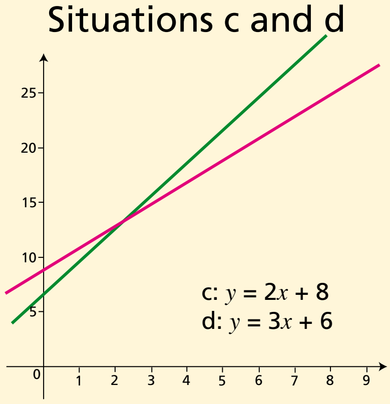
Situations a and b



Situations b and c



Situations c and d



Chapter 3: Proportional and non proportional situations

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Student Activity: Proportional and non proportional situations JCOL

In the following examples students examine instances of proportional and non proportional relationships for linear functions. Students arrive at a knowledge of the characteristics of a proportional relationship when it is graphed – it is linear and passes through the origin, has no 'start-up' value, i.e. no y-intercept and the 'doubling strategy' works. In other words, if x is doubled (or increased by any multiple) then y is doubled (or increased by the same multiple).

Example 1: The following table shows a pattern of growth of a fast growing plant.

Time in days	Height in cm
0	2
1	7
2	12
3	17
4	22
5	27
6	32

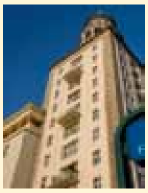
1. What information does the table give about the pattern of growth?
2. Can you predict the height of the plant after 12 days? Can you do this in different ways?
3. If the number of days is doubled will the height also double?
4. After 4 days is the height double what it was after 2 days? Explain your answer.
   After 6 days is the height double what it was after 3 days? Explain your answer.
   Did the 'doubling strategy' work?
5. What is the formula for the height of the plant in terms of the number of days elapsed and the start up value (use words first, then use symbols)?
6. Can you predict from the formula whether or not the 'doubling strategy' will work?

When the 'doubling strategy' does not work we are dealing with a non proportional situation. When the 'doubling strategy' works it is a proportional situation. The variables are proportional to each other.

Example 2: This table represents a context where each floor in a building has 5 rooms.

Number of floors	Total number of rooms
0	0
1	5
2	10
3	15
4	20
5	25
6	30

How many rooms in total would be in a building which has 12 floors?



1. Find 3 different ways to solve this problem.
2. If you double the number of floors what happens to the total number of rooms in the building?
3. If you treble the number of floors what happens to the total number of rooms in the building?
4. Does the ‘doubling strategy’ work?
5. Does the ‘trebling strategy’ work?
6. Are the variables proportional to each other?
7. What is the formula relating the number of rooms to the number of floors (use words first, then use symbols for HL)?
8. Can you predict from the formula, whether or not the ‘doubling strategy’ will work?

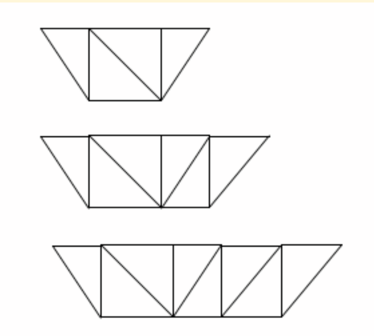
Plot graphs for the above 2 tables from example 1 and example 2

9. What is the same about the two graphs? What is different about the two graphs?
10. For which graph will the ‘doubling strategy’ work? Explain.
11. Which graph represents a situation where the variables are proportional to each other?

Example 3: In Home Economics students made gingerbread men. They used 2 raisins for the eyes and 1 raisin for the nose on each one. How many raisins did they use for 5 gingerbread men?

1. Represent this information in a table showing the number of raisins used for 0 to 5 gingerbread men.
2. State the number of raisins used in words.
3. State the number of raisins used in symbols.
4. Plot a graph to model the situation.
5. Does the 'doubling strategy' work here? Explain from the graph, table and symbols.

Example 4: Growing 'worm'



The diagrams show a new born, 1 day old, and 2 day old 'worm' made from triangles.

1. Draw up a table for the number of triangles used for a 0 to 5 day old worm.
2. What is the relationship between the age of the worm and the number of triangles used (use words first and then symbols)?
3. How many triangles in a 4 day old worm?
4. How many triangles in an 8 day old worm? Does the 'doubling strategy' work?
5. Are the variables proportional to each other? Explain using the table and formula.
6. Plot a graph to model the situation.
7. How does the graph show whether or not the variables are proportional to each

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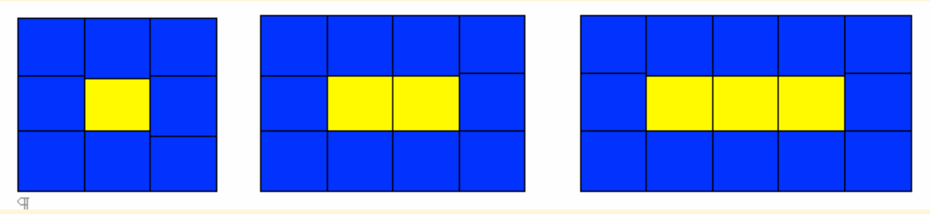
Example 5: How many blue tiles are needed for n yellow tiles?
(JC HL and may be appropriate for OL also.)



This activity has great potential and can be used with students at any stage. Students must be allowed to make the patterns themselves. There is a great opportunity to reinforce the concept of equality when students generalise the relationship differently. Students can make the patterns themselves with blocks or draw them. By physically making the pattern students will start to see the relationships.

1. Represent in a table the number of blue tiles needed for 0 to 5 yellow tiles.
2. How many blue tiles are needed for 4 yellow tiles? (accessible for OL)
3. What is the relationship between the number of blue tiles and the number of yellow tiles? Express this in words initially. (accessible for OL)
4. Consider how other students express this relationship. Are all the expressions equal? How can you verify this?
5. Does the doubling strategy work here?
6. If not why not?
7. Draw a graph showing the relationship between the number of yellow and the number of blue tiles. Can you explain from the table and the graph whether or not the 'doubling strategy' works?
8. Are the variables proportional to each other? Explain.

Class Discussion on proportional and non proportional situations

0	0
1	5
2	10
3	15
4	20
5	25
6	30

0	6
1	8
2	10
3	12
4	14
5	16
6	18

0	2
1	7
2	12
3	17
4	22
5	27
6	32

0	0
1	3
2	6
3	9
4	12
5	15
6	18

1. When you look at these tables how can you tell if the variables are in proportion?
2. Model the 4 situations represented in the tables by drawing graphs for each table.
3. When you look at linear graphs where the variables are proportional what do you notice about the graphs?
4. When you look at linear graphs where the variables are not proportional what do you notice about the graphs?
5. Write a formula relating the variables for each of these tables.
6. In the formulae there is an amount you multiply by and an amount you add. How are these quantities represented in the graphs?
7. Give a real life context for each of these tables.

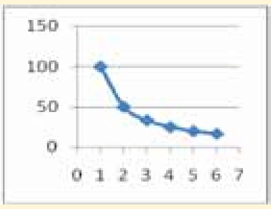
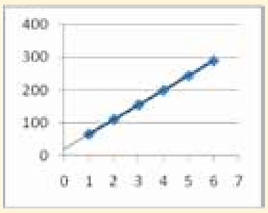
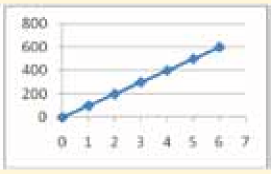
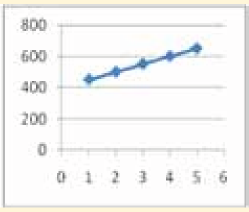
Next activity on graph matching:

JC HL: Matching all the different representations simultaneously

JC OL: Matching representations in appropriate pairs

Match up the stories to the tables, graphs, and formulae and fill in missing parts.

(This could form the basis for group or class discussion. Note these and other examples are referred to on the next page. Label the axes of the graph.)

<table>
<tr><th>Story</th><th>Table</th><th>Graph</th><th>Formula</th><th>Linear/
non linear,
proportional/non
proportional?
Justify</th></tr>
<tr><td>Changing euro to cent.</td><td><table><tr><th>No. of cars</th><th>€/w</th></tr><tr><td>1</td><td>450</td></tr><tr><td>2</td><td>500</td></tr><tr><td>3</td><td>550</td></tr><tr><td>4</td><td>600</td></tr><tr><td>5</td><td>650</td></tr></table></td><td>
</td><td>Y = 45x + 20</td><td></td></tr>
<tr><td>John works for €12 per hour. The graph and table show how much money he earns for the hours he has worked.</td><td></td><td>
</td><td>y=100/x</td><td></td></tr>
<tr><td>Salesman gets paid €400 per week plus an additional €50 for every car sold.</td><td><table><tr><th>m/ kg</th><th>t/ mins</th></tr><tr><td>1</td><td>65</td></tr><tr><td>2</td><td>110</td></tr><tr><td>3</td><td>155</td></tr><tr><td>4</td><td>200</td></tr><tr><td>5</td><td>245</td></tr><tr><td>6</td><td>290</td></tr></table></td><td>
</td><td>y=100x</td><td></td></tr>
<tr><td>A cookbook recommends 45 mins per kg to cook a turkey plus an additional 20 minutes.</td><td></td><td>
</td><td>y=12x</td><td>Linear and proportional because it's a line (constant change) through (0,0)</td></tr>
<tr><td>Henry has a winning ticket for a lottery for a prize of €100. The amount he receives depends on how many others have winning tickets.</td><td><table><tr><th>h</th><th>A/€</th></tr><tr><td>1</td><td>12</td></tr><tr><td>2</td><td>24</td></tr><tr><td>3</td><td>36</td></tr><tr><td>4</td><td>48</td></tr><tr><td>5</td><td>60</td></tr></table></td><td>
</td><td></td><td></td></tr>
</table>

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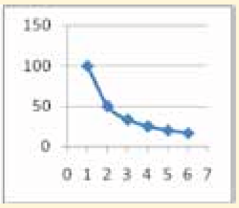
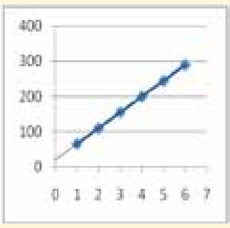
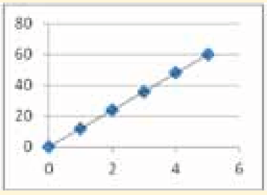
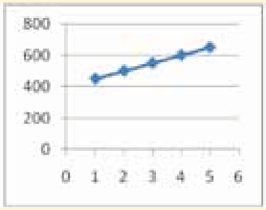
<table>
<tr><th>Story</th><th>Table</th><th>Graph</th><th>Formula</th><th>Linear/ non linear, proportional/non proportional? Justify</th></tr>
<tr>
<td>Henry is one of the winners of a prize of €100. The amount he receives depends on how many others have winning tickets.</td>
<td><table><tr><th>n</th><th>€</th></tr><tr><td>1</td><td>100</td></tr><tr><td>2</td><td>50</td></tr><tr><td>3</td><td>33.33</td></tr><tr><td>4</td><td>25</td></tr><tr><td>5</td><td>20</td></tr><tr><td>6</td><td>16.67</td></tr></table></td>
<td>
</td>
<td>y = 100/x
Can you write this formula another way?</td>
<td>Not linear.
(This relationship represents an inverse proportion. This is dealt with later on in the document.)
For each couple (x,y) in this situation what is the value of the product xy?</td>
</tr>
<tr>
<td>A cookbook recommends 45 minutes per kg to cook a turkey plus and additional 20 minutes.</td>
<td><table><tr><th>m/kg</th><th>t/k mins</th></tr><tr><td>1</td><td>65</td></tr><tr><td>2</td><td>110</td></tr><tr><td>3</td><td>155</td></tr><tr><td>4</td><td>200</td></tr><tr><td>5</td><td>245</td></tr><tr><td>6</td><td>290</td></tr></table></td>
<td>
</td>
<td>y = 20+45x</td>
<td>Linear but not proportional
The graph is a line but it does not pass through (0, 0)</td>
</tr>
<tr>
<td>John works for €12 per hour. The graph and table show how much money he earns for the hours he has worked.</td>
<td><table><tr><th>h</th><th>€</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>12</td></tr><tr><td>2</td><td>24</td></tr><tr><td>3</td><td>36</td></tr><tr><td>4</td><td>48</td></tr><tr><td>5</td><td>60</td></tr></table></td>
<td>
</td>
<td>y = 12x</td>
<td>Relationship is linear and proportional as its graph is a line through (0,0).</td>
</tr>
<tr>
<td>Changing euro to cent.</td>
<td><table><tr><th>euro</th><th>cent</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>100</td></tr><tr><td>2</td><td>200</td></tr><tr><td>3</td><td>300</td></tr><tr><td>4</td><td>400</td></tr><tr><td>5</td><td>500</td></tr></table></td>
<td>
</td>
<td>y = 100x</td>
<td>Relationship is linear and proportional because its graph is a line through (0,0). The table shows constant change for successive outputs where the input x is a natural number.</td>
</tr>
</table>

<table>
<tr><th>Story</th><th>Table</th><th>Graph</th><th>Formula</th><th>Linear/
non linear,
proportional/non proportional?
Justify</th></tr>
<tr>
<td>Fixed wage + commission</td>
<td><table>
<tr><th>Cars sold</th><th>€</th></tr>
<tr><td>0</td><td>400</td></tr>
<tr><td>1</td><td>450</td></tr>
<tr><td>2</td><td>500</td></tr>
<tr><td>3</td><td>550</td></tr>
<tr><td>4</td><td>600</td></tr>
<tr><td>5</td><td>650</td></tr>
</table></td>
<td>
</td>
<td>$y = 400 + 50x$</td>
<td>Linear but not proportional.
The graph is a line but it does not pass through (0,0).</td>
</tr>
</table>

Examples of proportional situations:

- Changing euro to cent – if you have no euro you have no cents, if you double the number of euro you double the number of cent, if you treble the number of euro you treble the number of cent Number of cent = 100 x number of euro.
- Changing miles to kms.
- If you get paid by the hour the more hours you work the more money you get in your wages.
- The circumference of a circle is proportional to its diameter (also to its radius).
- If you drive (walk, run or cycle) at a constant speed the distance travelled is proportional to the time taken.

Give 2 other examples of proportional situations.

Examples of non proportional:

- The recommended time it takes to cook a turkey is 45 minutes per kilogram plus an additional 20 minutes: total cooking time in minutes = 45 x number of kg + 20 minutes.
- You are a car salesman and you get paid a fixed wage of €400 per week + €50 for every car sold.
- When you pay your ESB bill you pay a fixed standing charge plus a fixed amount per unit of electricity used.
- The costs for a retailer selling shoes, for instance, are the costs of shoes purchased plus fixed overhead costs (rent, heating, wages, etc.)

Give 2 other examples of non proportional situations

What is the same and what is different about the following relationships?

(i) To convert metres to inches the approximate conversion factor is 1 metre equals 39.37 inches.

(ii) To convert temperatures from degrees Celsius to degrees Fahrenheit multiply the number of degrees Celsius by 1.8 (9/5) and add 32.

Chapter 4: Non constant rates of change-quadratic functions

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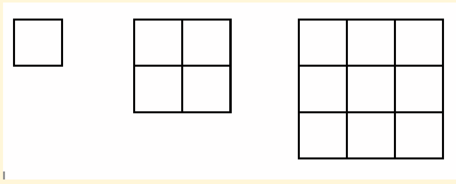
Student Activity: Non constant rates of change - quadratic functions JC OL

We saw that for linear relationships, which give straight line graphs, there was a constant rate of change as seen by constant change in the outputs for consecutive input values. Are all rates of change constant? We will look for patterns in different situations and investigate their rates of change. Students will investigate contexts which give rise to quadratic functions through the use of tables, graphs and formulae.

When dealing with graphs of quadratic functions students should contrast the quadratic with the linear functions as follows:

- the graphs are non-linear
- the graphs are curved
- the rate of change is not constant
- the rate of change of the rate of change is constant i.e. constant change of the changes
- the highest power of the independent variable is 2.

Example 1: Growing Squares Pattern. Draw the next two patterns of growing squares.



We wish to investigate how the number of tiles in each pattern is related to the side length of each square. Identify the independent and dependent variables.

Complete the table

Side length of each square	Number of tiles to complete each square
1	1
2	4
3	9
4	
5	
6	
7	
8	
9	
10	

1. How is the number of tiles used to make each square related to the side length of the square? Write the answer in words and symbols. What difference do you notice about this formula and the formulae for linear relationships?

2. Looking at the table, is the relationship linear? Explain your answer.

3. Predict what a graph of this situation will look like. Will it be a straight line? Explain your answer. Plot a graph to check your prediction.

4. What shape is the graph? Is the rate of change of the number of tiles constant as the side length of each square increases? Explain using both the table and graph.

5. Why is the graph not a straight line? How can you recognise whether or not a graph will be a straight line using a table?

Note: Students should come up with the last two columns in the table below themselves when looking for a pattern in the outputs and through questioning. It is not envisaged that they will be labelled as below initially for them.

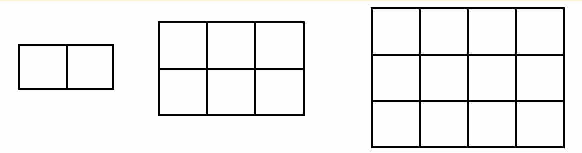
Side length	Number of tiles	Change	Change of changes
1	1	3	
2	4	5	2
3	9		
4			
5			
6			
7			
8			
9			

6. We saw that the changes in the table were not constant. Is there a pattern to them? Calculate the change of the changes.

7. When the change of the changes is constant we call this relationship between variables a quadratic relationship.
8. Give 3 characteristics of a quadratic relationship from the ‘growing squares’ example.

Example 2: Growing Rectangles JCOL

Complete the next two patterns in this sequence of rectangles.



Look at the pattern of growing rectangles. Make a table for the number of tiles in each rectangle for rectangles of heights from 1 to 10. Make observations about the values in the table.

What would a graph for this table look like? Will it be linear? How do you know? Make a graph to check your prediction.

Complete the table

Height of rectangle	Width of rectangle	Number of tiles to make each rectangle
1	2	2
2	3	
3		
4		
5		
6		
7		
h		

1. Can you see 1 group of 2 (or 2 groups of 1) in the first diagram, 2 groups of 3 (or 3 groups of 2) in the second diagram, etc?
2. The last row of the table gives a formula for the number of tiles in a rectangle of height h.
   Write down the formula in terms of the height of each rectangle.
   Call this formula (i)
   Check that the formula works for values already calculated.
3. Will a graph for this table be linear?
   Explain using the table below.
   Is the change of the changes constant?
   What type of relationship is indicated by the pattern of the changes? Explain
   What do you notice about the change of the changes?

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Height of rectangle	Number of tiles	Change	Change of changes
1	2		
2	6	4	
3			
4			
5			
6			
7			
h			

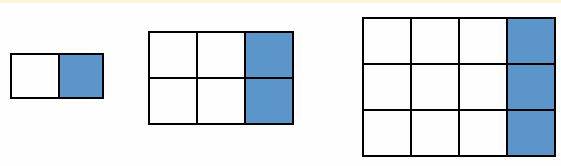
4. Predict what a graph of the situation will look like.
   Plot a graph to check your prediction.

5. What shape is the graph?

6. Does this relationship have the 3 characteristics identified in the last example for a quadratic relationship? Explain.

Relate growing rectangles to growing squares JC OL (with scaffolding)

Make the biggest square possible from each of the 5 rectangles you have previously drawn. Shade the remaining area in each rectangle. Draw the next two rectangles in the pattern below.



Complete the table for the number of tiles in each rectangle in terms of the height of each rectangle.

Height	**Number of tiles**	**Area of the rectangle = area of square + area of a rectangle**
1	2	1^2+1
2	6	
3	12	
4		
5		
6		
7		
h		

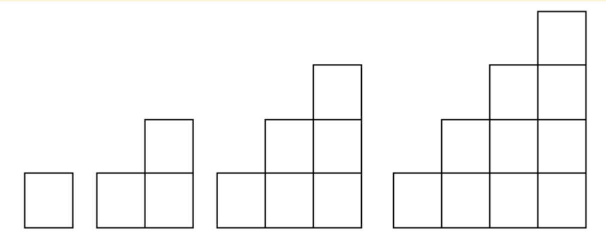
1. What is the formula for the number of tiles in a rectangle of height h in the above sequence, using the pattern seen in the right hand column with specific numbers?
   Call this formula (ii)

2. Show that the two formulae (i) and (ii) derived for the number of tiles in the each rectangle are equivalent expressions using (a) substitution and (b) the distributive law.

Example 3: Staircase Towers JC HL

Look at the staircases below. Make a table representing the relationship between the total number of tiles and the number of towers. Make observations about the values in the table. What would a graph look like? Would it be linear? How do you know? Make a graph to check your prediction.

Allow students to continue the pattern themselves and make their own observations. The following questions could be used as prompt questions if necessary.



1. How many tiles do you add on to make staircase 2 from staircase 1?
2. How many tiles do you add on to make staircase 3 from staircase 2?
3. How many tiles do you add on to make staircase 4 from staircase 3?
4. Complete the table below

Number of towers	Number of tiles
1	1
2	3
3	6
4	
5	
6	
7	
8	
9	
10	

5. Will a graph for this table be linear? Explain, using the table below.

Side length	Number of tiles	Change	Change of changes
1	1		
2	3	2	
3	6	3	1
4			
5			
6			
7			

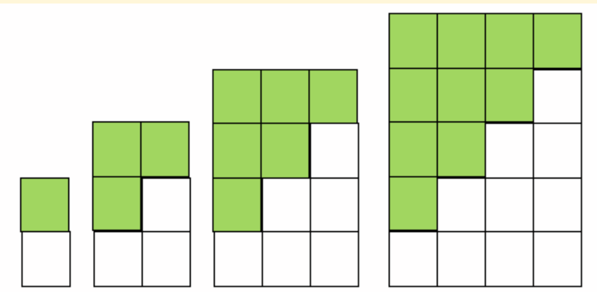
6. Are the changes constant? Are the changes of the changes constant?

7. Predict what a graph of the situation will look like.
   Plot a graph to check your prediction.

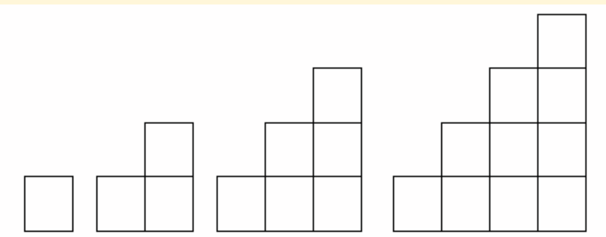
8. What shape is the graph?

We need to develop a formula for the number of tiles required for the nth tower.

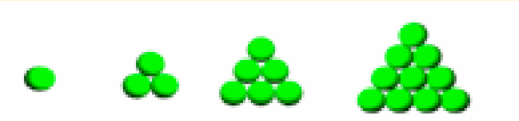
Compare the pattern for the 'growing rectangles' with that of the staircase towers.



1. What is the formula for the number of tiles in the nth rectangle in terms of n?



2. Hence what is the formula for the number of tiles in the nth staircase in terms of n?



3. The staircase pattern showing the numbers 1, 3, 6, 10, ... could also be shown using the above pattern. What do you think these numbers are called? (Look at the shapes formed).

Contrast the rates of change for ‘growing rectangles’ and ‘staircase towers’ – use tables and graphs JC HL

Growing rectangles

Side length	Number of tiles	Change	Change of changes
1	2	4	2
2	6	6	2
3	12	8	
4	20	10	
5	30		
6			
7			
8			
9			
10			

Staircase Towers

Side length	Number of tiles	Change	Change of changes
1	1	2	1
2	3	3	
3	6		
4			
5			
6			
7			

Compare how the values of y increase for the same change in x in ‘growing rectangles’ and ‘staircase towers’. Focus on how the numbers change in the first two columns

1. What is the same about the graphs and the tables for ‘growing rectangles’ and ‘staircase towers’?

2. What is different about the graphs and the tables for ‘growing rectangles’ and ‘staircase towers’?

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3. How is the non-linear nature of the graph related to the tables and the context?
4. What do the changes increase by for ‘growing rectangles’?
5. What do the changes increase by for ‘staircase towers’?
6. How is the difference in the answers to the last two questions shown in the graphs for each situation?
   Which situation shows the greatest change of the changes?

Example 4: The growth of a fantasy creature called Walkasaurus JC HL

The following table shows how Walkasaurus’s height changes with time.

Age (years)	Height (cm)
0	1
1	2
2	4
3	7
4	11
5	16
6	22

1. Will a graph for this table be linear? Explain using a table.
   Note: The last two columns of the table need not be given to students. Students should draw on the skills they have learned in previous activities and should not need these prompts. The aim of this approach is to ask questions that give students the opportunity to think and investigate in order to make sense of the features of a quadratic relationship.

Age (years)	Height (cm)	Change	Change of changes
0	1	1	
1	2	2	1
2	4		
3	7		
4	11		
5	16		
6	22		

2. Predict what a graph of the situation will look like.
   Plot a graph to check your prediction.

3. What shape is the graph?
   Compare the pattern of y values for Walkasaurus's height with the pattern of y values for the staircase towers.
4. How is the pattern similar?
5. How is the pattern different?
6. By comparing with the staircase towers pattern write a formula for Walkasaurus's height h in terms of his age d in years.
7. Using the same axes and scales plot the graphs for the staircase towers and Walkasaurus's height?

   What is the same and what is different about the two graphs?

Example 5: Gravity and Quadratics JCHL

This question is set in context so that students will see the application of quadratics in everyday life.

a. If a cent is dropped from a height of 45m, its height changes over time according to the formula $h = 45 - 4.9\ t^2$, where t is measured in seconds. Use mathematical tools (numerical analysis, tables, graphs) to determine how long it will take for the cent to hit the ground.
   What does the graph of height vs. time look like?
   What connections do you see between the graph and the table?
b. Suppose you wanted the cent to land after exactly 4 seconds. From what height would you need to drop it? Explain your answer.
c. Suppose you dropped the cent from the top of the Eiffel Tower (300m high).
   How long would it take to hit the ground?
   What does the graph of height vs. time look like?
   What connections do you see between the graph and the table?

JCHL Linear and quadratic:

i Investigate the relationship between length and width for a rectangle of fixed perimeter

ii Investigate how change in the dimensions of a rectangle of fixed perimeter affects its area.

Given a rectangle with a fixed perimeter of 24 metres, if we increase the length by 1m the width and area will vary accordingly. Use a table of values to look at how the width and area vary as the length varies.

Length/m	Width /m	Area/m^2
0	12	0
1	11	11
2	10	
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		

1. Predict the type of graph you expect to get if you plotted width against length?
   Explain your answer using the values of the change between successive outputs.
2. What is different about these changes and changes in previous examples, e.g. sunflower growth?
   Plot the graph of width against length to confirm your predictions.
3. What can you say about the slope of this graph?
4. Explain what the slope means in the context of this situation.
5. Express the width in terms of the length and perimeter of the rectangle.
6. What are the variables and the constants in the situation?
7. Does it matter which you decide to call the independent variable?
8. What do you notice about the values of area in the table – are they increasing, decreasing or is there any pattern?
9. Does the area appear to have a maximum or minimum value?
   If it does what value is it?
10. Predict the type of graph you expect to get if you plotted area against length. Explain your answer.
11. Plot the graph of area against length to confirm your predictions.
12. What shape is this graph?
13. Use the graph of area plotted against length, for a fixed perimeter of 24, to find which length of rectangle gives the maximum area.
    Does this agree with your conclusion from Q 9?

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14. Express the area in terms of the length and perimeter of the rectangle. Check that the formula works by substituting in values of length from the table.
    Extension: What if the rectangle had a fixed area – what is the relationship between the length and width? (See inverse proportion).

Chapter 5: Exponential growth

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Student Activity: Exponential growth

JCOL LCOL, LCHL, – Exponential relations limited to doubling and tripling

Students investigate situations involving exponential functions using words, tables, graphs and formulae and understand that exponential functions are expressed in constant ratios between successive outputs.

Students should see applications of exponential functions in their everyday lives and appreciate the rapid rate of growth and decay shown in exponential functions.

Example 1: Pocket money story

Contrast adding 2 cent each day (multiplication as repeated addition) with multiplying by 2 each day (exponentiation as repeated multiplication).

I put this proposition for pocket money to my Dad at the beginning of July.

"I just want you to give me pocket money for this month. All I want is for you to give me 2c on the first day of the month, double that for the second day, and double that again for the 3rd day ... and so on. On the 1st day I will get 2c, on the 2nd day 4c, on the 3rd day 8c and so on until the end of the month. That is all I want."
Was this a good deal for Dad or is it a good deal for me?

Make a table to show how much money I will get for the first 10 days of the month.

Time/days	Money/c
0	2
1	4
2	
3	
4	
5	
6	
7	
8	
9	
10	

1. Using the table would you expect a graph of this relationship to be linear? Explain.
2. Using the table would you expect a graph of this relationship to be quadratic? Explain.
3. Check changes, change of the changes, change of the change of the changes etc.
4. What do you notice? Is there a pattern to the differences? If so what is it?
5. Predict the type of graph you will get if you plot money in cent against time in days.
6. Make a graph to check your prediction. What do you notice?
7. Can you come up with a formula for the amount of money you will have after n days?
8. Using your calculator find out how much money you would get on the 10th day, the 20th day, the 30th day, 31st day?
9. What are the variables in the situation? What is constant in this situation?
10. Where is the factor of 2 (the doubling) in the table? Where is it in the graph?
11. Contrast this situation with adding 2c every day. How much would you have on day 31?
12. How would the formula change if your Dad trebled the amount of money, starting by giving you 2c on the first day as above? Make a table and come up with the formula for this new situation.
13. When people speak of 'exponential growth' in everyday terms, what key idea are they trying to communicate?

$y=ab^x$

Final value = starting value multiplied by (growth factor) $^{\text{no. of intervals of time elapsed}}$

(with the independent variable in the exponent).

This type of growth is called exponential growth. The variable is in the exponent. There is a constant ratio between successive outputs. This constant is called the growth factor. During each unit time interval the amount present is multiplied by this factor. Can you think of other examples of things which grow in this way e.g. double over a constant period of time? (The growth factor does not have to be a 'doubling'.)

Computing: Byte = 2^3 bits, kilobyte = 2^{10} bytes, Megabyte = 2^{20} bytes

Population growth, paper folding, cell division and growth, compound interest – (repeated multiplication)

Exponential growth in legend–the origins of the game of chess and the payment for its inventor

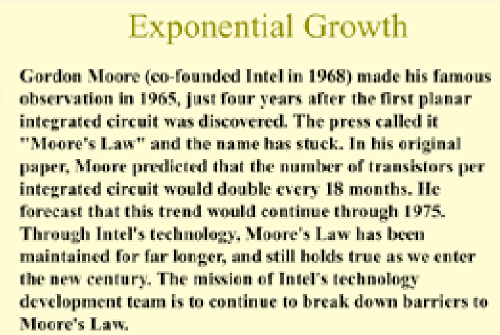
Legend tells us that the game of chess was invented hundreds of years ago by the Grand Vizier Sissa Ben Dahir for King Shirham of India. The King loved the game so much that he offered his Grand Vizier any reward he wanted. "Majesty, give me one grain of wheat to go on the first square of the chess board, two grains to place on the second square, four grains to place on the third square, eight on the fourth square and so on until all the squares on the board are covered". The King was astonished. "If that is all you wish for you poor fool then you may have your wish". When the 41st square was reached, over one million million grains of rice were needed. Sissa's request required more than the kingdom's entire wheat supply. In all, the king owed Sissa more than 18,000,000,000,000,000,000 grains of wheat which was worth more than his entire kingdom. Foolish king!

Square	Rice on each square	Pattern for rice on the square	Rice on the board
1	1	1	1
2	2	1x2	1+2
3	4	1x2x2	1+2+4
4	8		
5			

Exponential growth and computing power

See the following link for more on this legend and Moore's Law dealing with the exponential growth of transistors in integrated circuits leading to smaller and faster computers.

http://www.authorstream.com/Presentation/Arkwright26-16602-wheat-1-Exponential-Growth-Moores-Law-seedcount-presentation-002-Entertainment-ppt-powerpoint/



Example 2: How many ancestors do you have? JC OL

1. You have 2 parents, 4 grandparents, and eight great grandparents. Going back further into your family tree, how many ancestors do you have 5 generations ago, n generations ago? Make a table to show this. (30 years on average per generation?)
2. What example is this the same as numerically? How does it differ from it? (Think in terms of time intervals.)

Example 3: Paper folding JC OL

If I fold a sheet of paper in half once I have 2 sections. If I fold it in half again I have 4 sections. What happens if I continue to fold the paper? Make a table, plot a graph and come up with a formula to describe the situation. Is it linear, quadratic, exponential or none of these?

Example 4: Growth of bacteria JC HL

1. If we start with 1 bacterium which by growth and cell division becomes 2 bacteria after one hour – in other words the number of bacteria doubles every hour – how many of these bacteria would there be at the end of 1 day (24 hours)?
2. What type of growth is this – linear, quadratic, exponential, or none of these?
3. Would this growth increase indefinitely? Explain. (Exponential growth would not continue indefinitely – the bacteria would run out of for example nutrients or space.)
4. Certain bacteria double in number every 20 minutes. Starting with a single organism with a mass of 10-12g, and assuming temperature and food conditions allowed it to grow exponentially for 1 day, what would be the total mass produced?
5. What would be its total mass after 2 days? (The mass of the earth is approximately 6 x 1024kg.)

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Example 5: Compound interest and exponential growth LC OL

€650 is deposited in a fixed interest rate bank account. The amount (final value) in the account at the end of each year is shown in the following table.

End of year	1	2	3	4	5
Final value	€676.00	€703.04	€731.16	€760.41	€790.82

1. How can you tell from the table if the above relationship is linear, quadratic or exponential? Explain.
2. If you plot a graph of final value against time what does the graph look like for this limited range of times? How might plotting more points help?
3. What would be the effect of increasing the interest rate? Make a table showing the final values for the first five years using an interest rate of 10% per annum compound interest. Plot a graph for this data. Compare the graph to the graph produced for the lower interest rate.
4. What formula expresses the final value after t years in the above compound interest formula if the initial value is €650?
5. What is the danger of drawing a conclusion about a relationship from a plot using a limited number of points?
6. From just the 2 data points below what are two possible formulae which could relate the x and y values?

x	y
1	2
2	4

Population growth

The tendency of populations to grow at an exponential rate was pointed out in 1798 by an English economist, Thomas Malthus, in his book *"An Essay on the Principle of Population"*. He suggested that unchecked exponential growth (of course people die and war and famine still occur) would outstrip the supply of food and other resources and would lead to disease and wars. He wrote that "Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in arithmetic ratio".

Of course population growth for any particular country depends on the following factors: birth rate, death rate, immigration, emigration, wars, famine, diseases and other possible factors.

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Example 6: Growth of world's population LC OL

1. The world's population reached 6 billion in 1999. On 07/01/2009 the world's population was 6,755,987,239. http://www.census.gov/ipc/www/popclockworld.html
   On 07/01/10 the world's population was 6,830,586,985. (Write each of these figures to 3 significant figures if you wish and use scientific notation.) Approximately how many people more were in the world one year after 07/01/09?
   What factor did the world's population increase by in the year? This is the growth factor (rounded to 2 decimal places).
2. Assuming this growth rate continues what do you multiply the population in 2010 by to get the population in 2011?
3. Make a table of expected world population values from 2010 to 2015. Note how much the population increases by each year.
4. What type of growth is this?
5. Write a formula for the expected value of the world's population n years from 2010?
6. Using trial and error and the expected growth rate of 1% per year, how many years from 2010 would you expect the world's population to have doubled?
7. Using trial and error and the expected growth rate of 2% per year, how many years from 2010 would you expect the world's population to have doubled?
8. (LC HL) Using logs work out the exact number of years for the world's population to double assuming a constant growth rate of 1% per year.

http://www.learner.org/interactives/dailymath/population.html

The rate of Earth's population growth is slowing down. Throughout the 1960s, the world's population was growing at a rate of about 2% per year. By 1990, that rate was down to 1.5%, and by the year 2015, it's expected to drop to 1%. Family planning initiatives, an aging population, and the effects of diseases such as AIDS are some of the factors behind this rate decrease.

Even at these very low rates of population growth, the numbers are staggering. By 2015, despite a low expected 1% growth rate, experts estimate there will be 7 billion people on the planet. By 2050, there may be as many as 10 billion people living on Earth.
Can the planet support this population?
When will we reach the limit of our resources?

Example 7: Population of Sylvania LC HL

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The table shows the population in thousands of a small mythical country for various years

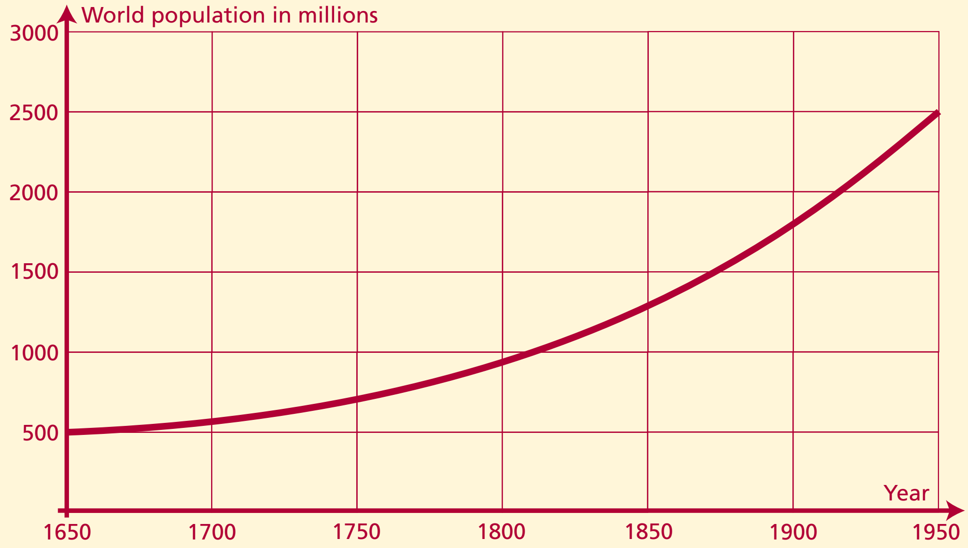
Year	Population in thousands
1825	200
1850	252
1875	318
1900	401
1925	504
1950	635
1975	800

1. Using the table above, check to see if the relationship between time and population is linear, quadratic or exponential or none of these? Explain.
2. Predict what a graph of the data will look like.
3. Plot a graph of this data. What does the graph look like?
4. About how long does it take for the population to double? Explain how you got the answer from the table. Explain how you got the answer from the graph.
5. What would you expect the population to be in 2000, 2050, and 3000? Explain how you got these answers from (i) the table (ii) the graph
6. Here are 3 different formulae that can be used to calculate population in this situation.

   $P = 200(2)^{\frac{t}{75}}$ $\quad P = 200(2)^{\frac{(r-1825)}{75}}$ $\quad P = 401(2)^{\frac{(n-1900)}{75}}$

   Explain what t, r, and n represent.
7. Explain how $P = 200(2)^{\frac{t}{75}}$ fits the table and the situation and how it can yield the correct answer.
8. Explain how $P = 200(2)^{\frac{(r-1825)}{75}}$ fits the table and the situation and how it can yield the correct answer.
9. Explain how $P = 401(2)^{\frac{(n-1900)}{75}}$ fits the table and the situation and how it can yield the correct answer.
10. Do you expect actual data values to match data values obtained using the formula? Explain.

Example 8 LC HL Is the population growth shown in the graph below exponential?



1. Make a table and investigate if the growth is exponential – does it always increase by the same factor for successive equal time intervals?
   Calculate the ratio between output values for successive equal time intervals.

2. Calculate what constant growth factor over the first 100 years (1650 to 1750) would have given the same result in that period.
   Use this growth factor for the next two intervals of 100 years.
   Plot the points for 1850 and 1950.
   What can you conclude about this growth pattern?

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Exponential Decay: Compounded depreciation (an investment decreases to 0.80 of its value at the end of every year) and the decay of nuclear waste are examples of exponential decay. They involve repeated multiplication but in this case the growth factor is less than 1.

Example 1 LC OL Alan is 4 metres away from a wall. He jumps towards the wall and with each jump he halves the distance between himself and the wall.

Draw up a table of values to show Alan's distance from the wall with each step (jump).

Step	Distance from the wall/m
0	4
1	2
2	
3	
4	
5	
6	

1. Looking at successive output values in the table, explain whether or not the relationship is linear, quadratic, or exponential.
2. Plot a graph of Alan's distance from the wall against the step number towards the wall.
3. How is this graph different than the graph obtained for compound interest?
4. How is this graph like the graph obtained for compound interest?
5. Will Alan's distance from the wall ever be zero?

Example 2: LC OL During a visit to the hospital, Jane receives a dose of radioactive medication which decays or loses its effectiveness at a rate of 20% per hour.

1. If she receives 150mg of the medication initially, approximately how many milligrams of the medication remain in her body after the first 6 hours?
2. Draw up a table showing the amount of radioactive medication in her body for 1 up to 6 hours, increasing by 1 hour each time.
3. Plot a graph of mg of medication left in her body against hours elapsed.
4. Is it possible to reduce the amount of radioactive medication in Jane's body to 0? Explain.
5. What are the implications of this for the decay of radioactive waste material? (Some radioactive materials take more than 20,000 years for half of it to decay. See mathematical tables for half lives of radioactive elements.)

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Chapter 6: Inverse Proportion

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Student Activity: Inverse Proportion

For an input x and an output y, y is inversely proportional to x, if there exists some constant k so that $xy = k$.

The constant k is called the constant of proportionality.

Example 1: Sharing a fixed amount of prize money

Imagine you won the lotto and the prize was €1,000,000. You don't know how many other people also had the winning numbers so you are speculating on how much you will collect. If you are the only winner then you collect €1,000,000. However, if there are 2 winners you collect €500,000.

1. Make up a table showing number of winners and the corresponding amount won per person.
2. Is it a linear relationship? Explain
3. Is it a quadratic relationship? Explain.
4. Plot a graph of prize collected per winner against number of winners.
5. Describe the graph.
6. Is this like exponential growth/decay, i.e. is there a constant ratio between successive outputs for equal changes of the independent variable?
7. What formula relates winnings per person, number of winners and total value of the prize?
8. There are variables and a constant in the equation. Identify the variables and the constant.
9. Which variable is the independent variable?
10. Which variable is the dependent variable?

    *(The resulting graph is called a **hyperbola**. In real life contexts we are only considering positive values of the independent variable.)*

11. As the number of winners doubles what happens to the amount of money each winner gets?
12. As the number of winners trebles what happens to the amount of money each winner gets?

13. As the number of winners quadruples what happens to the amount of money each winner gets?
14. When the number of winners decreases what happens to the amount of money each person wins?
15. When the number of winners becomes very big what happens to the amount of money each person receives
16. Will the graph ever touch the x axis? Explain.
17. Will the graph ever touch the y axis? Explain.

This type of relationship is called an **inverse proportion**. The product of the variables is a constant.

- The more people who share a pizza the smaller the slice of pizza each one receives. The size of the slice is inversely proportional to the number of people sharing.
- Length varies inversely with width for a rectangle of given area. $A = l \text{ x } w$.
- Depth of a fixed volume of liquid in different cylinders varies inversely as the area of the base of the cylinder. V = Area of base x h

Example 2: Relationship between time spent travelling a fixed distance and speed for the journey

1. Consider a situation where people travel between two points a fixed distance apart. Assume that different people travel with different constant speeds between the two points. What happens to the time taken to complete the journey between the two points as the speed increases?
2. Is there a speed which will give a time of 0? Explain.
3. Is there a value of time corresponding to a speed of 0? Explain.
4. What do these two answers tell you about the graph?
5. Make up a table showing speed and the corresponding time taken. (You can choose the distance between the two points and the speed and the units used.)
6. Is it a linear relationship between speed and time? Explain.
7. Is it a quadratic relationship? Explain.
8. Plot a graph of time taken against speed.
9. Describe the graph.
10. Is this like exponential growth/decay – is there a constant ratio between successive outputs for equal time intervals?

11. What formula relates the speed, time taken and the distance between the two points?
12. There are variables and a constant in the equation. Identify the variables and the constant
13. Which variable is the independent variable?
14. Which variable is the dependent variable?

*The resulting graph is called a **hyperbola**.*

15. As the speed doubles what happens to the time taken for the journey?
16. As the speed trebles what happens to the time taken for the journey?
17. As the speed quadruples what happens to the time taken for the journey?
18. When the speed has a low value what happens to the time taken for the journey?
19. When the speed has a high value what happens to the time taken for the journey?
20. Will the graph ever touch the x axis? Explain.
21. Will the graph ever touch the y axis? Explain.
22. Is this an example of an inverse proportion? Explain.
23. Give two examples of inverse proportions.

http://www.articlesforeducators.com/dir/mathematics/cat_and_mouse.asp

If it takes 5 cats 5 days to catch 5 mice, how long will it take 3 cats to catch 3 mice?

If a boy and a half can mow a lawn in a day and a half, how long will it take 5 boys to mow 20 lawns?

Hint: Only vary 2 quantities at any one time – keep the third one constant.

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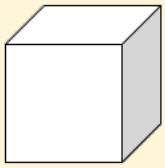
Chapter 7: Moving from linear to quadratic to cubic functions

Student Activity:Moving from linear to quadratic to cubic functions. LC OL

The linear model, occurs when there is a constant rate of change. When there is a consistent force for change, the quadratic model often fits well. For example, gravity is a constant force. Many movements under its influence are essentially quadratic. Cubic models appear in locations such as highway designs which require a smooth transition from a straight line into a curve.

Example 1: Using a cube students investigate linear, quadratic and cubic relationships



For a block with edge lengths of 1 unit, the perimeter of the base is 4 units, the surface area is 6 square units and the volume is 1 cubic unit. What would the values be for a block with edge lengths of 2 units or 3 units or 34 units or n units?

1. Make tables for perimeter, for surface area and for volume as the edge lengths of the block increase. Examine the tables to predict the shape of the graph for each of the three relationships. Explain your predictions. Make the graphs for perimeter vs. edge length, surface area vs. edge length and volume vs. edge length and compare them with your predictions.

Students complete a table for the perimeter of the base for cubes with different edge lengths

Edge length /cm	Perimeter of the base of the cube /cm
1	
2	
3	
4	
5	
6	
7	

1. Predict the shape of graph for the above relationship.
2. Explain your prediction.
3. Write a formula for the perimeter of the base in terms of the edge length.

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4. Plot a graph to show the above relationship.
5. Check if values for perimeter predicted by the formula agree with values predicted by the graph.

Complete the table below for total surface area of the cube.

Edge length /cm	Surface area of the cube / cm^2
1	
2	
3	
4	
5	
6	
7	

1. Predict the shape of graph for the above relationship.
2. Explain your prediction.
3. Write a formula for the total surface area in terms of the edge length.
4. Plot a graph to show the above relationship.

Complete the table below for volume of the cube

Edge length/cm	Volume/cm^3	Change	Change of the changes	Change in the changes of the changes
1	1			
2	8	7	12	1
3	27	19		
4				
5				
6				
7				

6. Predict the shape of graph for the above relationship.
7. Explain your prediction.
8. Write a formula for the perimeter of the base in terms of the edge length.
9. Plot a graph to show the above relationship.
10. Check if values for perimeter predicted by the formula agree with values predicted by the graph.

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Relationship between surface area and volume for a cube.

Do you think the volume and surface area of a cube will ever be equal numerically?

Do you think the volume will be ever be numerically greater than the surface area?

Check by completing the following table.

Edge	1	2	3	4	5	6	7	8	9	n
Area of base										
Total surface area										
Volume										
$\dfrac{\textbf{Surface Area}}{\textbf{Volume}}$										

At what point is the ratio of surface area to volume equal to 1?

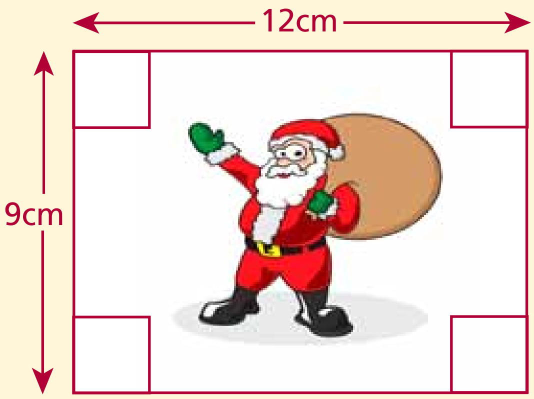
When is volume less than surface area?

When is volume greater than surface area?

How can you explain the rapid growth of volume and the slower growth of surface area?

Example 2 (LC HL) Designing the largest box – cuboid shaped – from a rectangular sheet of fixed dimensions

You are making boxes from cardboard for personalised Christmas presents. You wish to put as many sweets as possible into the boxes using rectangular based boxes but you are limited by the size of the sheets of cardboard available. (Alternative: design a suitcase to hold maximum volume)



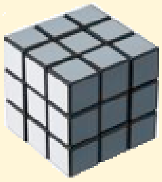
Start with a rectangular sheet of cardboard 9cm x 12cm. From each corner cut squares of equal size. Fold up the four flaps and tape them together forming an open box. Depending on the size of the squares cut from the rectangular sheet, the volume of the box will vary. Investigate how the volume of the box will vary.

(The student will engage in problem solving using their experiences from previous activities. One of the aims of this approach is to empower students to use the multiple representations to help solve problems and some representations may be more useful than others. Below are suggested prompt questions where students are having difficulty.)

1. When the box is formed which dimension of the box will be represented by x?
2. If we cut out squares of side x from the cardboard, predict what value of x will give you the largest volume box.
3. Write a formula for the volume V of the box in terms of the length of the box l, its width w, and height x.
4. Use a table to test out values of x to see which value will give the box of largest volume. Check them using the formula.

x/cm	l/cm	w/cm	V/cm
0.0			
0.5			
1.0			
1.5			

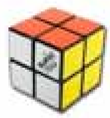
5. Between what two x values do you expect the largest volume to be?
6. Do you expect a graph of V against x to be linear, quadratic or cubic? Explain your answer. Plot the graph of V against x.
7. From the graph approximately what value of x will yield the maximum volume?



Example 3 The painted cube (LC HL)

You have a cube like the one shown on the right, but it has been thrown into a bucket of paint. The outside is covered with paint but the inside is not. You break it apart into unit cubes to see how many of the unit cubes have 0 faces painted, 1 face painted, 2 faces painted, 3 faces painted , 4 faces, 5 faces, or 6 faces painted.

You are looking for a pattern between the number of painted faces and the original size of the cube. Explore this for a 2x2x2 cube up to a 6x6x6 cube. For each cube, count the number of blocks in each of the categories 0 faces painted, 1 face painted, 2 faces painted, 3 faces painted, 4 faces, 5 faces, or 6 faces painted.(Students can come up with the table themselves as they should be used to doing this as a problem solving strategy.)

Dimensions	0 faces painted	1 face painted	2 faces painted	3 faces painted	4 faces painted	5 faces painted	6 faces painted	Total number of unit blocks
2x2x2 	0	0	0	8	0	0	0	8
3x3x3 								
4x4x4 			24		0			64
5x5x5 								
6x6x6 		96						
nxnxn (n>=2)								n^3

Verify the equivalent expression for n^3 by adding up all the expressions in columns 2 – 8 of the last row.

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Example 4 (extension for HL)

Find a relationship between total surface area (TSA) of a cube and the perimeter (P) of one side.

Edge length/cm	P of one side	TSA
1	$4(1) = 4$	$6(1)^2 = 6$
2	$4(2) = 8$	$6(2)^2 = 24$
3		
4		
5		
6		
n		

1. Predict what you expect a graph of total surface area plotted against perimeter of a cube to look like. Explain your prediction.
2. Using the formulae for perimeter and total surface area, derive a formula for total surface area in terms of perimeter.
3. Using the formula derived check that it correctly predicts values of total surface area given the perimeter values in the table.

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Chapter 8: Appendix containing some notes and suggested solutions

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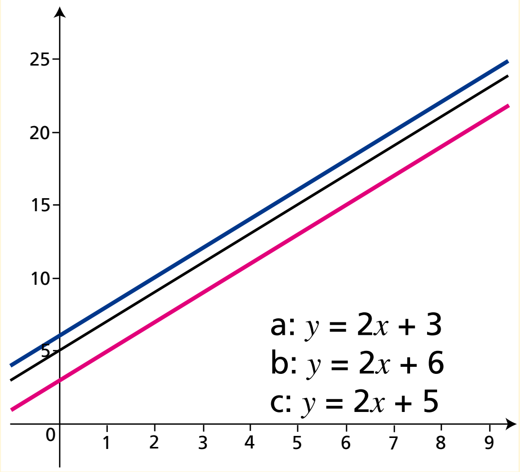
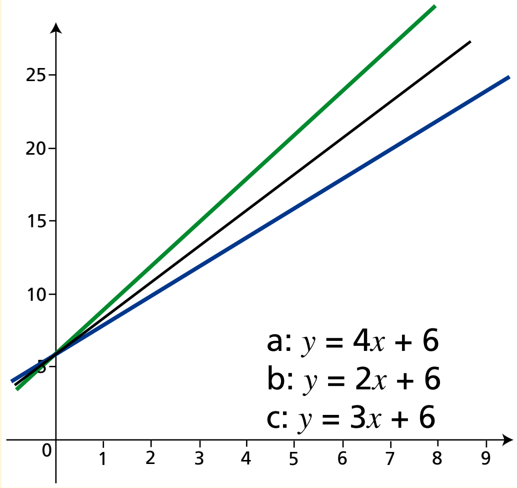
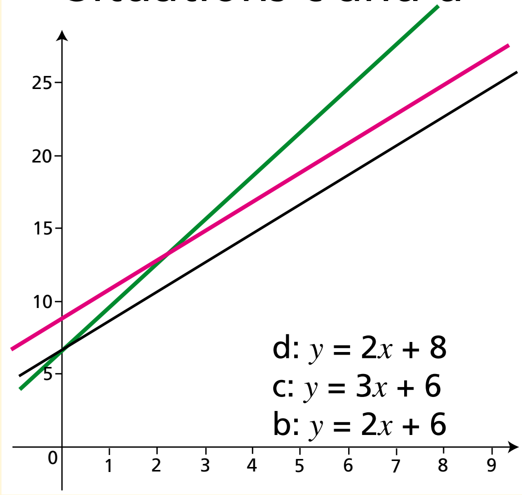
Situations a and b



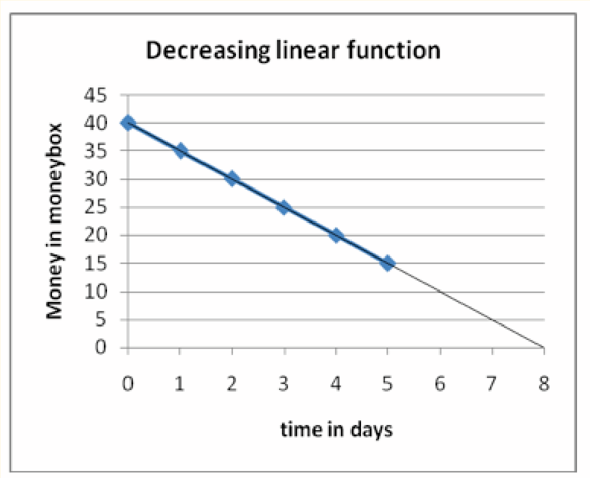
Situations b and c



Situations c and d



Graphs for growing rectangles, staircase towers, and Walkasaurus



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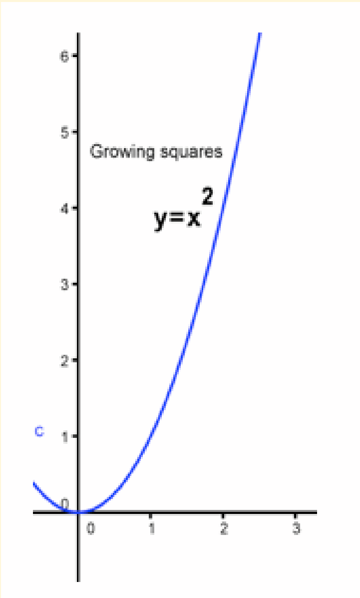

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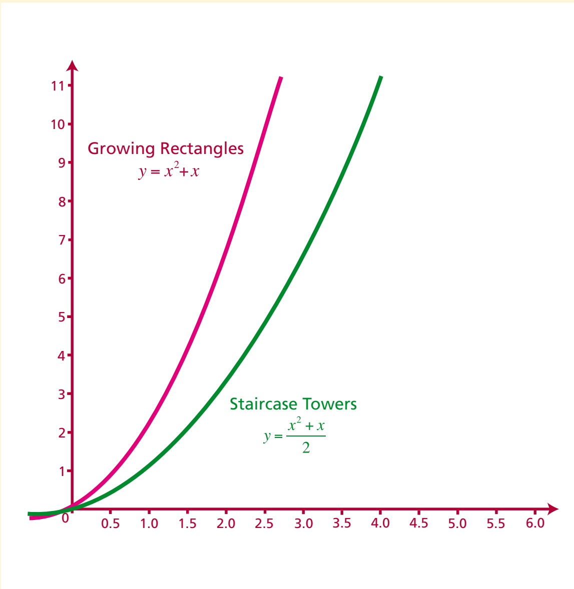

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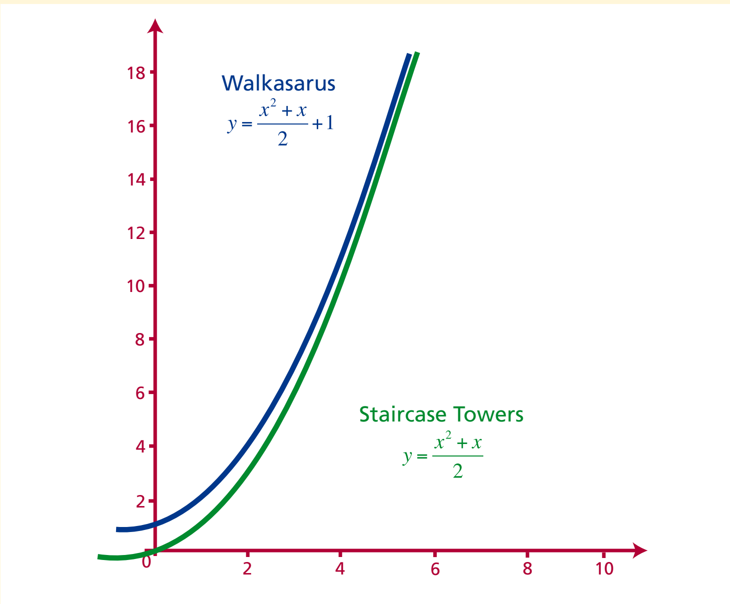

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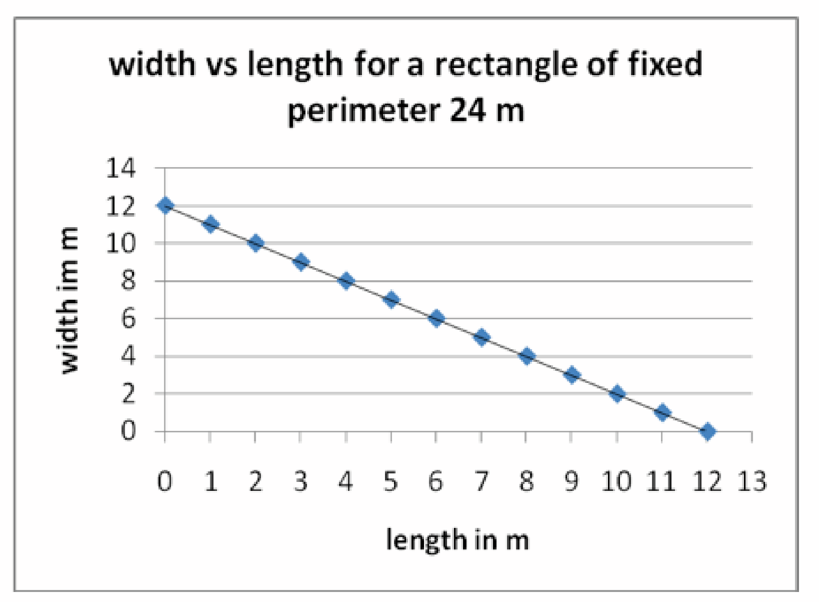
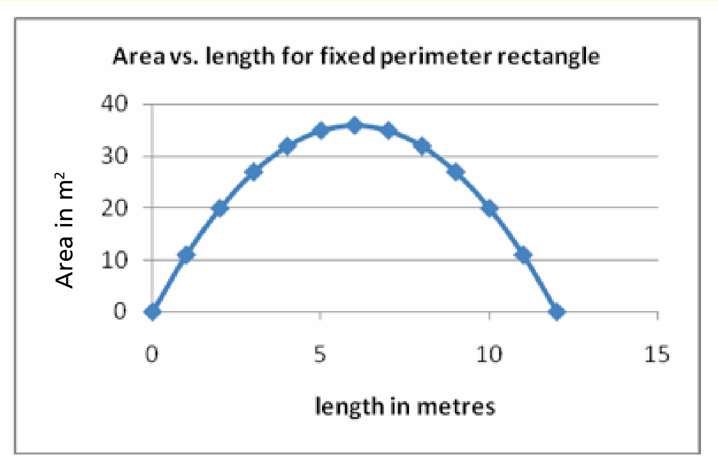
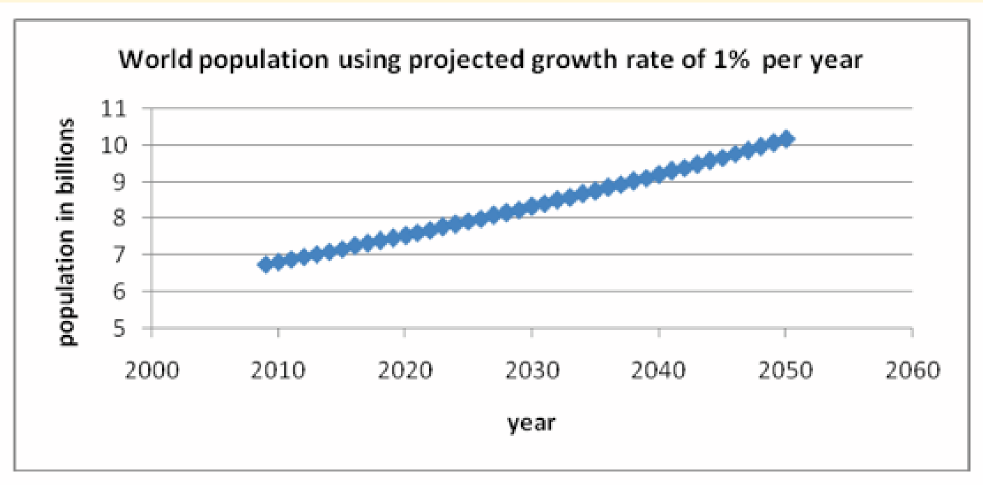


Graphs for Student Activity Page 38



Graphs for Student Activity Page 38

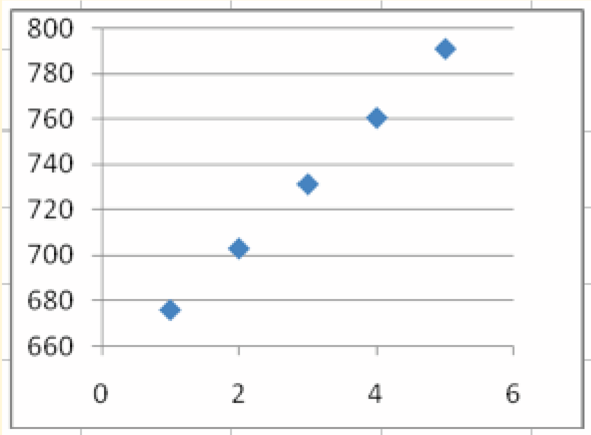


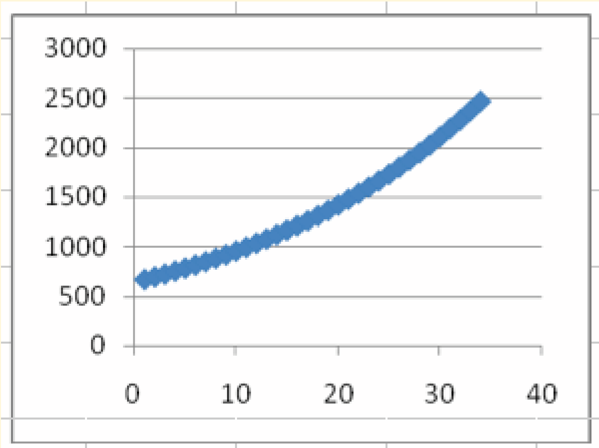
Graphs for Student Activity Page 47

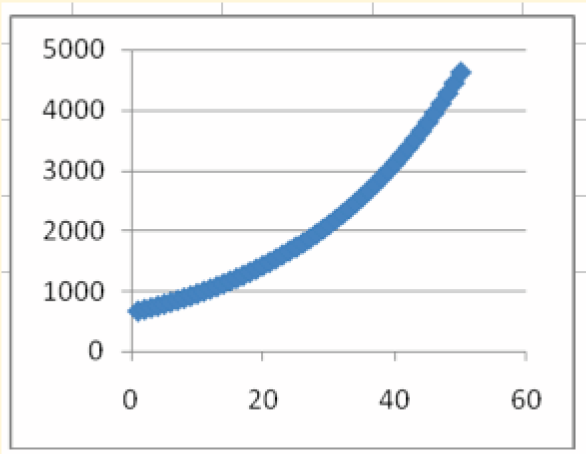
Graphs for Student Activity Page 46

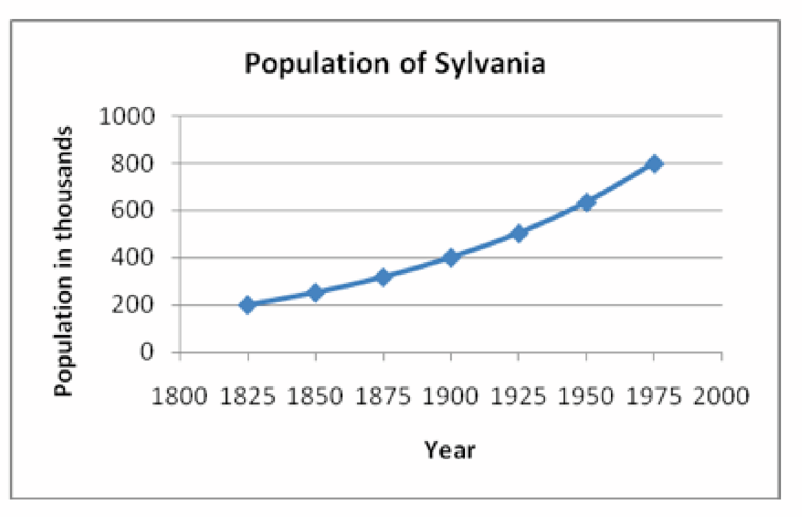
Spreadsheet for Example 6 Compound Interest

1	676
2	703.04
3	731.16
4	760.41
5	790.82
6	822.4528
7	855.3509
8	889.5649
9	925.1475
10	962.1534
11	1000.64
12	1040.665
13	1082.292
14	1125.583
15	1170.607
16	1217.431









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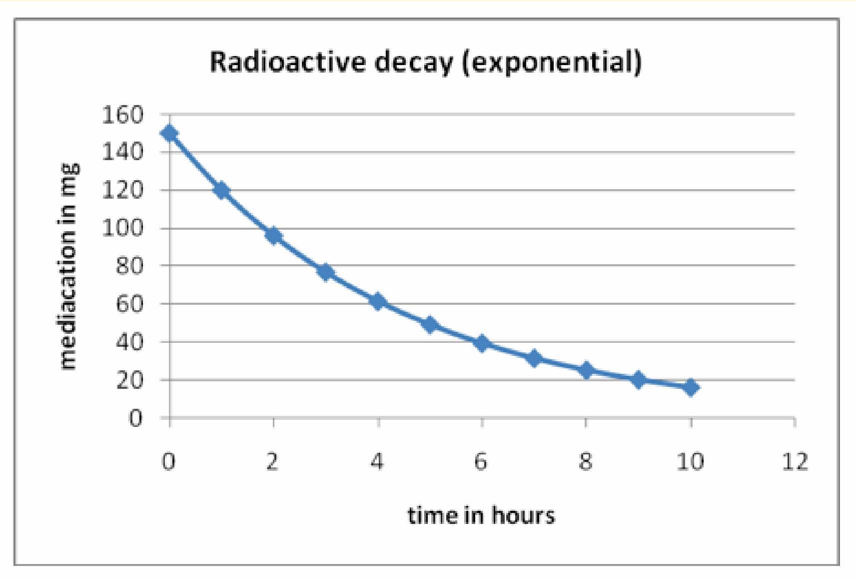

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Graphs for Student Activity Page 54

Graphs for Student Activity Page 59
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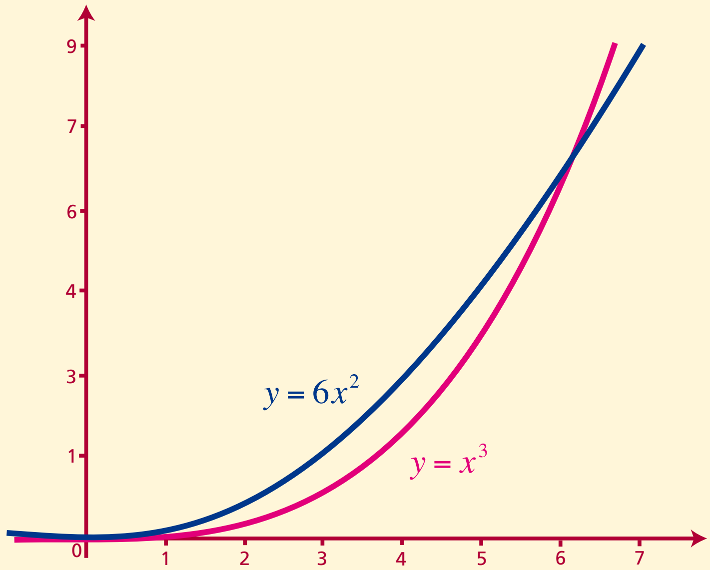
Showing growth of surface area and growth of volume for a cube of side x



Implications for faster growth rate of volume compared to surface area

Surface area is important as it effects how fast an object cools off (greater area equals quicker cooling. Machines that need to get rid of extra heat have little metal fins stuck to them to increase their surface area).

Babies have a large surface area compared to their volume and lose heat quickly – hence they must be wrapped up very well to prevent heat loss. As they grow the ratio of surface area to volume decreases (volume grows faster than surface area) and so they do not lose heat as quickly.

Surface area affects how quickly a droplet evaporates – greater area gives faster evaporation.

Surface area affects how quickly a chemical reaction proceeds – greater area gives faster reaction.

Insects breathe through their surface area (x^2) but their need for oxygen is in proportion to their volume (x^3) – hence they can't become the size of humans as their surface area would not supply enough oxygen for such a large volume.

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Why can't giants exist?

Consider an adult woman 160cm tall and a giant woman 10 times as tall. How tall is the giant? 16m

Assume the giant is similar to the woman in composition of bones, flesh etc. If the woman weighs 50kg, what does the giant weigh? (Weight, like volume, increases as height^3)

Giant weighs 50 x 10 x 10 x 10 = 50,000kg

The woman's weight is supported by her two leg bones. If the cross sectional area of each of her leg bones is 10cm^2, how much weight is supported by each cm^2 of leg bone?

50/20 kg/cm^2 = 2.5 kg/cm^2

What is the cross sectional area of each of the giant's bones?

10 x 10^2 = 1,000cm^2

How much weight is supported by each cm^2 of the giant woman's leg bones?

50,000/2000 = 25 kg/cm^2

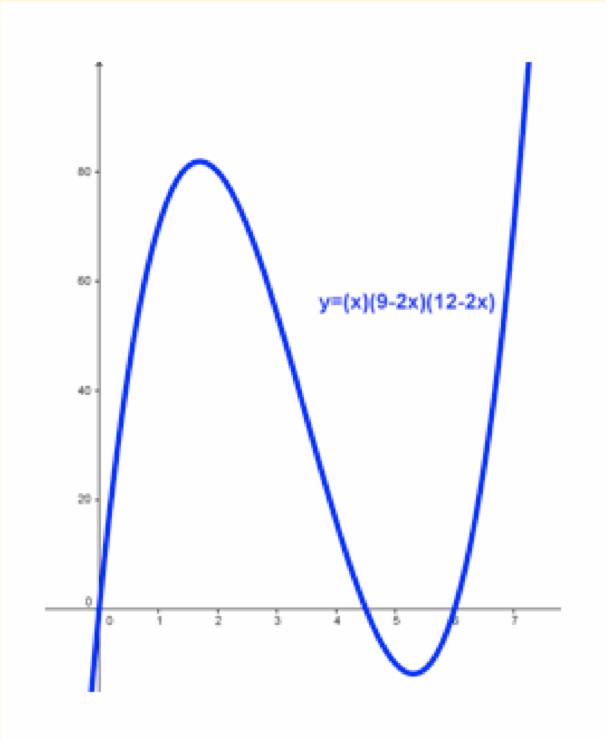
If the giant's bones are the same density as those of the woman of average height, what will happen to the giant's leg bones when she stands up?

They will break because they have 10 times the weight to carry for the same surface area.

How could this problem be sorted? Denser bones for the giant, or walk on all fours to distribute the weight.

Graphs for Student Activity Page 60

Making a box with the largest volume, given a rectangle of fixed dimensions



x	width	length	Volume
0	9	12	0
0.5	8	11	44
1	7	10	70
1.5	6	9	81
2	5	8	80
2.5	4	7	70
3	3	6	54
3.5	2	5	35
4	1	4	16
4.5	0	3	0
5	-1	2	-10
5.5	-2	1	-11
6	-3	0	0
6.5	-4	-1	26

The Painted Cube, page 62

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Dimensions	0 faces painted	1 face painted	2 faces painted	3 faces painted	4 faces painted	5 faces painted	6 faces painted	Total number of unit blocks
2 x 2 x 2 	0	0	0	8	0	0	0	8
3 x 3 x 3 	1	6	12	8	0	0	0	27
4 x 4 x 4 	8	24	24	8	0	0	0	64
5 x 5 x 5 	27	54	36	8	0	0	0	125
6 x 6 x 6 	64	96	48	8	0	0	0	216
nxnxn (n>=2)	$(n-2)^3$	$6(n-2)^2$	$12(n-2)$	8	0	0	0	n^3

$$\left.\begin{array}{lcl} (n-2)^3 & = & n^3-6n^2+12n-8 \\ 6\,(n-2)^2 & = & 6n^2-24n+24 \\ 12\,(n-2) & = & 12n-24 \\ +8 & = & 8 \end{array}\right\} \quad \boxed{\text{Total } n^3}$$

Explanation:

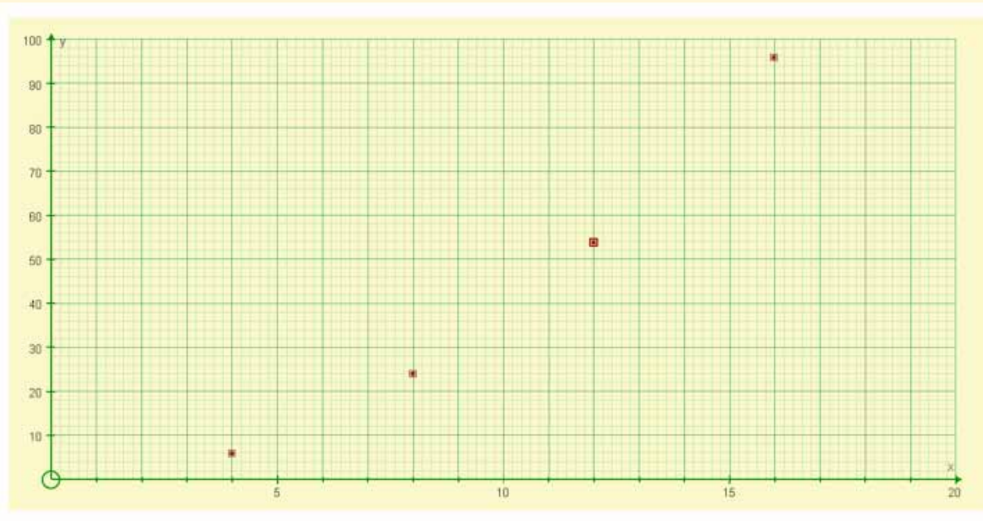
For the 2 x 2 x 2, no cubes are painted on all faces
The cubes with 3 painted faces are always on the corners and there are 8 of those.
The cubes with 2 painted faces occur on the edges between the corners. There are 12 of those if it is a 3 x 3 cube and 12 $(n-2)$ if an n by n cube (take away the 2 on the corners).
The cubes with 1 painted face occur as squares on each of the 6 faces and there will be 6 $(n-2)^2$ of those.
Those with no painted faces are on the inside and are given by $(n-2)^3$

Suggested Solution, Student Activity page 68

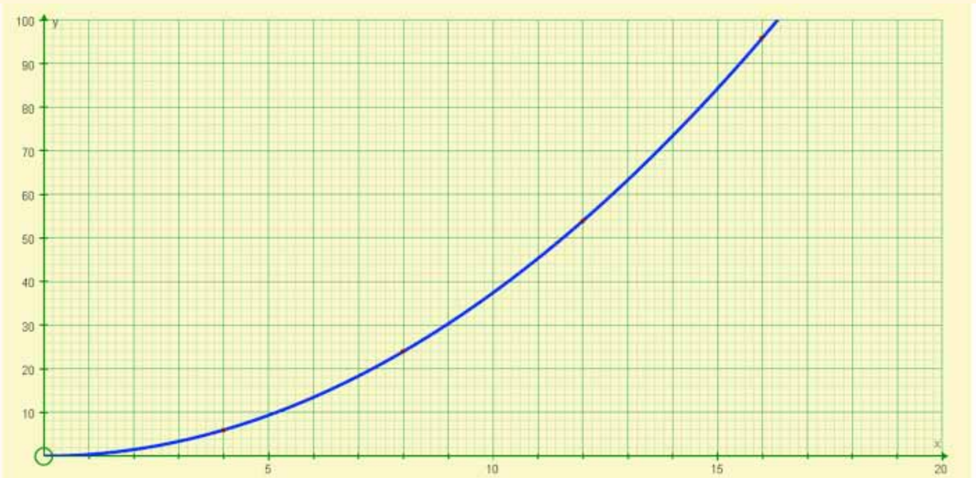
Relationship between TSA and Perimeter of a face for a cube

	P of one side	TSA	Couples
1cm	4(1)	6(1)2	(4, 6)
2cm	4(2)	6(2)2	(8, 24)
3cm	4(3)	6(3)2	(12, 54)
4cm	4(4)	6(4)2	(16, 96)
$n$cm	4(n)	6(n)2	

Relationship between TSA and perimeter of a cube.



Graph going through above set of couples



The equation of this graph is $y = 0.375x^2$.

I then asked the question "Is it possible for the students to derive the equation of the quadratic from the table before the graph is drawn with the couples?" The following approach works.

1. We are trying to find an equation which links Perimeter (P) and Total Surface Area (TSA).
2. From the table (the Tn)

   $P = 4n$ and $TSA = 6n^2$

   $$\left(n = \frac{P}{4} \quad and \quad n = \sqrt{\left(\frac{TSA}{6}\right)}\right)$$

   $$\frac{P^2}{16} = \frac{TSA}{6}$$

   $$\frac{6P^2}{16} = TSA$$

This is the equation of the quadratic graph $y = 0.375x^2$